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Single-objective Optimization using Chance-constrained Model: A Comparison Result Between Linear and Normal Uncertainty Distributions

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ABSTRACT: *Uncertain optimization problem is an important tool in the field of optimization and are widely used in real-world applications. In this paper, we consider an uncertain single-objective optimization problem with uncertainty in the coefficients of both objective function and constraints. We formulate the proposed problem and show that the corresponding chance-constrained optimization model can be converted into a deterministic model by considering linear and normal uncertainty distributions. This paper mainly aims to provide the solution techniques for an uncertain single-objective optimization problem and show a comparison between the optimial objective values in relation to uncertainty levels, under linear and normal uncertainty distributions. Finally, two numerical examples are presented to illustrate the usefulness of the proposed method.*
KEYWORDS: *Uncertain optimization, single-objective optimization, chance-constrained optimization, linear uncertainty distribution, normal uncertainty distribution.*

1 INTRODUCTION:

Optimization is a fundamental concept in operations research used extensively in several disciplines, such as engineering, economics, and finance. For traditional optimization, in the literature, many researchers have worked so far on developing effective methodologies and algorithms to solve problems associated with exact and deterministic coefficients of the objective and constraint functions. However, in most real-world optimization problems such as finance, logistics, and energy management, uncertainty is inherent where coefficients are often uncertain, meaning that they are not exactly known, which may lead significant impact on the reliability, effectiveness of the solutions and decrease decision-making quality [1, 2, 4-6]. The reasons for this data uncertainty includes computational and estimation errors, imprecise data, or lack of informations. In such case, to obtain solutions that remain feasible and optimal, the decision maker must tackle the uncertainty.

Various uncertainty optimization techniques, such as robust optimization [1-3, 7, 14], stochastic programming [4-6, 8, 15, 16], and distributionally robust optimization [9, 10] have been developed to address this challenge, and provide more reliable and resilient solutions. These powerful techniques have been effectively applied in various fields, including finance [11], logistics [12], and energy systems [13, 15].

Stochastic programming was established to deal uncertainty and solve uncertainty related problems. Avriel and Wilde [17] developed stochastic concept for geometric programming problem with deterministic exponents and coefficients that are nonnegative random variables. Shiraz et al. [18] used sensitivity analysis to investigate the stability and robustness of a chance-constrained Data Envelopment Analysis(DAE) model with random-rough data inputs and outputs. The authors considered a chance-constrained DEA model with random data and solved it by proposing a deterministic equivalent model with quadratic constraints.

Uncertainty theory is a recent scope of research developed solely by Liu[19] and extensive work has been done by the author on some research problems in uncertainty theory [20] and for solving uncertain geometric programing problems [21]. Charnes et al.[22, 23] used chance-constrained techniques via uncertainty theory for solving critical path analysis. Shiraz et al. [24] developed geometric programming model using uncertainty theory for problems with the objective and constraint coefficients that follow normal, linear, and zigzag uncertainty distributions. Solano-Charris et al. [25] used unconventional techniques such as the Non-dominated Sorting Genetic Algorithm (NSGAII) and the Multiobjective Evolutionary Algorithm (MOEA) to solve bi-objective robust vehicle routing problem with uncertain costs and demands. For single-objective stochastic optimization numerous methods have been put forward in the literature to overcome the difficulties posed by uncertainty. One popular approach via stochastic programming involves modeling the uncertain parameters using probability distributions and solving the resulting optimization problem [5, 8]. Stochastic gradient descent and sample average approximation are also popular methods used to solve uncertain single-objective stochastic optimization problems by Powell [26].

From the literature survey, we see that a lot of study has been done on typical optimization problems where the coefficients in the objective and constraint functions are precise. However, real-world problems not involve precise coefficients. For solving uncertainty affected problems via stochastic approach, only geometric programming problem are considered in particular to use linear, normal, and zigzag uncertainty distributions. Therefore, we seek for the scope to study on the uncertainty-affected single objective optimization problems using chance-contrained approach where the the coefficients of the objective functions and constraints are assumed to be uncetrian variables(UV) that follow linear and normal uncertainty distributions and show a comparison between the corresponding optimal results.

- We formulate the uncertain single-objective optimization problems that include uncertain variables in the coefficients of objective and constraint functions.
- The equivalent chance-constrained optimization problem is then developed under linear and normal uncertainty distributions.
- After that, we show the convertion of uncertain chance-constrained models into a deterministic one and solve in a conventional way.
- Numerical examples and comparison between the corresponding optimal results are presented to validate our proposed method.

In order to thoroughly investigate the aforementioned problem carefully, the remainder of this paper is organized as follows. Section2 is intended to introduce some basic notions and theorems of uncertainty theory and concepts of uncertain chance-constrained programming. Section3 describes the formulation of uncertain single-objective optimization in a stochastic way. In Section4, we consider the chance-constrained single-objective optimization problem under linear and normal uncertainty distributions and show how the uncertain optimization problem can be converted into deterministic model. To show the efficiency of the approach, two numerical examples are presented in Section5 illustrating the comparison results of optimal objective values between linear and normal uncertainty distributions. Finally, we cover some conclusions in Section 6.

2 PRELIMINARIES

This section addresses some basic notions and major concepts of uncertainty theory and introduces uncertain chance-constrained programming, which will be used in the next sections. All concepts and definitions pertaining to uncertainty theory presented here are due to [19].

2.1 Uncertainty Theory

Definition 2.1 *Let Σ be the σ-algebra on a universal set X. Then a set function $\mu : \Sigma \to [0,1]$ is called an uncertain measure if it satisfies the following axioms.*

A1. $\mu(X) = 1$ for the universal set X. (Normality)

A2. $\mu(\mathsf{E}) + \mu(\mathsf{E}^c) = 1,$ for any $\mathsf{E} \subseteq X$. (Self-Duality)

A3. For all countable collection of events $\{\mathsf{E}_k\}_{k=1}^{\infty}$, $\mu\left(\bigcup_{k=1}^{\infty} \mathsf{E}_k\right) \le \sum_{k=1}^{\infty} \mu(\mathsf{E}_k).$

Definition 2.2 *Let Σ is a $\sigma -$ algebra on X and μ is an uncertain measure. Then the triplet (X, Σ, μ) is said to be an uncertainty space.*

Definition 2.3 *A measurable function $\xi : (X, \Sigma, \mu) \to \mathsf{R}$ is called an uncertain variable(UV), where R is the set of all real numbers.*

Definition 2.4 *An UV is said to be positive iff $\mu(\xi \le 0) = 0$ and nonnegative iff $\mu(\xi < 0) = 0.$*

Proposition 2.5 *Let $\xi_1, \xi_2, \ldots, \xi_n$ are UVs and f is a real-valued measurable function. Then the function $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an UV.*

Definition 2.6 *Let $\xi_1, \xi_2, \ldots, \xi_n$ be UVs on (X, Σ, μ). Then, for every event $\mathsf{E} \in \Sigma$,*

$$(\xi_1 + \xi_2 + \ldots \xi_n)(\mathsf{E}) = \xi_1(\mathsf{E}) + \xi_2(\mathsf{E}) + \ldots + \xi_n(\mathsf{E}) \text{ and } (\xi_1 . \xi_2 ... \xi_n)(\mathsf{E}) = \xi_1(\mathsf{E}) . \xi_2(\mathsf{E}) ... \xi_n(\mathsf{E}).$$

Definition 2.7 *The uncertainty distribution (UD) for the UV* ξ *is a function* $\Phi_\xi : \mathbb{R} \to [0,1]$ *defined by* $\Phi_\xi(x) = \mu(\xi \le x) \quad \forall x \in \mathbb{R}$.

Definition 2.8 *An UV* ξ *is called linear, if it has a linear UD and mathematically it is defined by,*

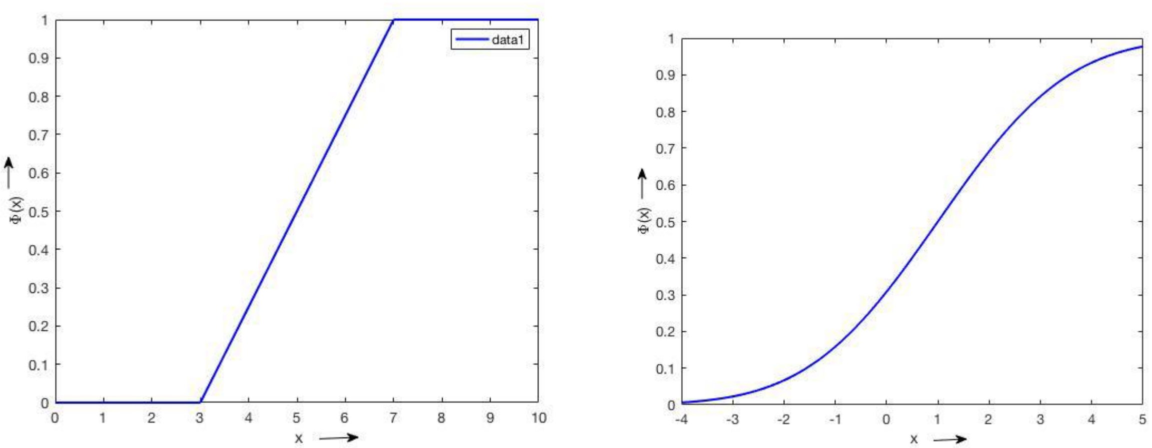
$$\Phi_\xi(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \ge b \end{cases}$$

where $a, b \in \mathbb{R}$ with $b > a$. An UV that follows linear UD is denoted as $\xi : L(a,b)$.

Definition 2.9 *An UV* ξ *is called normal, if it follows a normal UD. Mathematically this is defined by,*

$$\Phi_\xi(x) = \left[1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right]^{-1}, \forall\, x \in \mathbb{R}$$

where $e, \sigma \in \mathbb{R}$ and $\sigma > 0$. An UV that follows normal UD is denoted as $\xi : N(e,\sigma)$.



Linear UD Normal UD

Figure 1: Uncertainty distributions

Definition 2.10 *Let* ξ *be an UV. The expected value of* ξ *is defined by,*

$$E[\xi] = \int_0^{+\infty} \mu(\xi \ge s)ds - \int_{-\infty}^{0} \mu(\xi \le s)ds,$$

provided that atleast one of the two integrals is finite.

Proposition 2.11 *If* $\xi_1, \xi_2 \ldots, \xi_n$ *are UVs and* f *is real valued function of* $\xi_1, \xi_2 \ldots, \xi_n$*, then* $f(\xi_1, \xi_2 \ldots, \xi_n)$ *is a UV.*

Theorem 2.1 *Let* $\xi_1, \xi_2 \ldots, \xi_n, \lambda$ *be independent UVs on the space* (Υ, Σ, m) *with UDs* $\Phi_{\xi_1}, \Phi_{\xi_2}, \ldots, \Phi_{\xi_n}, \Phi_\lambda$ *respectively. If* $f(x, \xi_1, \xi_2 \ldots, \xi_n)$ *is a constraint function strictly increasing with respect to* $\xi_1, \xi_2 \ldots, \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2} \ldots, \xi_n$*, then for* $\delta \in (0,1)$,

$$m\{f(x,\xi_1,\xi_2\ldots,\xi_n)\le\lambda\}\ge\delta\equiv f(x,\Phi_{\xi_1}^{-1}(\delta),\Phi_{\xi_2}^{-1}(\delta),\ldots,\Phi_{\xi_m}^{-1}(\delta)),\Phi_{\xi_{m+1}}^{-1}(1-\delta)),\Phi_{\xi_n}^{-1}(1-\delta))\le\Phi_{\lambda}^{-1}(1-\delta)$$

.

Theorem 2.2 *Let* $\xi_1,\xi_2\ldots,\xi_n,\lambda$ *be independent UVs on the space* (Υ,Σ,m) *with UDs* $\Phi_{\xi_1},\Phi_{\xi_2},\ldots,\Phi_{\xi_n},\Phi_{\lambda}$ *respectively. If* $f(x,\xi_1,\xi_2\ldots,\xi_n)$ *is a constraint function strictly increasing with respect to* $\xi_1,\xi_2\ldots,\xi_m$ *and strictly decreasing with respect to* $\xi_{m+1},\xi_{m+2}\ldots,\xi_n$ *, then for* $\delta\in(0,1)$*,*

$$m\{f(x,\xi_1,\xi_2\ldots,\xi_n)\ge\lambda\}\ge\delta\equiv f(x,\Phi_{\xi_1}^{-1}(1-\delta),\Phi_{\xi_2}^{-1}(1-\delta),\ldots,\Phi_{\xi_m}^{-1}(1-\delta),\Phi_{\xi_{m+1}}^{-1}(\delta)),\Phi_{\xi_n}^{-1}(\delta))\ge\Phi_{\lambda}^{-1}(1-\delta)$$

2.2 Uncertain Chance-constrained Programming

Consider an uncertain optimization problem,

$$\begin{aligned}\min\quad & f(x,\xi)\\ \text{s.t.}\quad & h(x,\xi)\le\lambda,\end{aligned}\tag{2.1}$$

where x is the decision vector, ξ is the uncertain vector and $c(x,\xi)$ represents the uncertain constraints of the model. In the model, the constraints $h(x,\xi)$ do not provide a deterministic feasible solution, hence, it is preferred for these constraints to be hold with a chance δ , where δ is a predefined uncertainty level and is normally assumed to take values from the open interval $(0,1)$. Consequently, one can use the uncertain measure μ to express a chance constraint as,

$$m\{h(x,\xi)\le\lambda\}\ge\delta.$$

3 PROBLEM FORMULATION

A conventional deterministic single objective optimization problem of minimum type can be written as,

$$\begin{aligned}\min\quad & f(x)\\ \text{s.t.}\quad & h_j(x)\le\lambda_j,\quad j=1,2,\ldots,m,\end{aligned}\tag{3.1}$$

where $x\in\mathbb{R}^n$ is the decision vector, $f:\mathbb{R}^n\to\mathbb{R}$ is the objective function and h_j are the constraints of the optimization problem.

We write any function $f:\mathbb{R}^n\to\mathbb{R}$ splitting into n components as,

$$f(x)=\xi_1\psi_1(x)+\xi_2\psi_2(x)+\ldots+\xi_n\psi_n(x)=\sum_{k=1}^{n}\xi_k\psi_k(x).\tag{3.2}$$

Thus for each j , $h_j:\mathbb{R}^n\to\mathbb{R}$ admits the following form,

$$h_j(x)=\xi_{j1}\psi_{j1}(x)+\xi_{j2}\psi_{j2}(x)+\ldots+\xi_{jn}\psi_{jn}(x)=\sum_{k=1}^{n}\xi_{jk}\psi_{jk}(x),\ j=1,2,\ldots,m.\tag{3.3}$$

Hence accordingly the problem (3.1) can be reformulated as,

$$\begin{aligned}\min\quad & f(x)=\sum_{k=1}^{n}\xi_k\psi_k(x)\\ \text{s.t.}\quad & h_j(x)=\sum_{k=1}^{n}\xi_{jk}\psi_{jk}(x)\le\lambda_j,\quad j=1,2,\ldots,m,\end{aligned}\tag{3.4}$$

where $x\in\mathbb{R}^n$ is the decision vector, ξ_k and ξ_{jk} are the coefficients of the corresponding functions, and $\lambda_j(j=1,2,\ldots,m)$ are positive constants.

The deterministic optimization problem (3.4) can be converted into an uncertain optimization problem by introducing uncertainty in the coefficients ξ_k , ξ_{jk} and λ_j of the objective functions and the constraints respectively. Therefore, the converted uncertain optimization problem can be written as,

$$\begin{aligned} \min \quad & \sum_{k=1}^{n} \tilde{\xi}_k \psi_k(x) \\ \text{s.t.} \quad & \sum_{k=1}^{n} \tilde{\xi}_{jk} \psi_{jk}(x) \le \tilde{\lambda}_j, \quad j = 1,2,\ldots,m. \end{aligned} \tag{3.5}$$

where $x \in \mathbb{R}^n$ is the decision vector, $\tilde{\xi}_k, \tilde{\xi}_{jk}$ are UVs and the constraints model the impact of uncertainty in the problem.

A variant of the uncertain chance-constrained optimization model can be developed that takes into account the minimum expectation with the measuriblity of constraints. For each δ, where δ is a predefined uncertainty level, the equivalent uncertain optimization problem can be written as,

$$\begin{aligned} \min \quad & E\left[\sum_{k=1}^{n} \tilde{\xi}_k \psi_k(x)\right] \\ \text{s.t.} \quad & m\left[\sum_{k=1}^{n} \tilde{\xi}_{jk} \psi_{jk}(x) \le \tilde{\lambda}_j\right] \ge \delta, \quad j = 1,2,\ldots,m. \end{aligned} \tag{3.6}$$

In the following section, we propose a solving method for the uncertain optimization problem (3.6) considering the cases where the coefficients are assumed to be UVs with linear and normal UDs.

4 DETERMINISTIC CHANCE-CONSTRAINED MODEL

In this section, we develop an uncertain single-objective optimization model whose corresponding chance-constrained counterpart admits an equivalent crisp formulation, where the uncertain coefficients $\tilde{\xi}_k, \tilde{\xi}_{jk}$ and $\tilde{\lambda}_j$ in equation (3.6) are assumed to be UVS that follow the linear and normal UDs.

4.1 Uncertain Optimization with Normal Uncertainty Distributions

Let the coefficients $\tilde{\xi}_k, \tilde{\xi}_{jk}, \tilde{\lambda}_j$ in (3.6) be independent normal UVs with distributions characterized by positive parameters. That is, $\tilde{\xi}_k : N(\xi_k, \sigma_k), \tilde{\xi}_{jk} : N(\xi_{jk}, \sigma_{jk})$ and $\tilde{\lambda}_j : N(\lambda_j, \sigma_j)$, where $\xi_k, \xi_{jk}, \lambda_j, \sigma_k, \sigma_{jk}, \sigma_j$ are all positive real numbers. We now use the following lemma and a theorem to convert the model in equation (3.6) into a deterministic program and hence solve the program.

Lemma 4.1 *([19]) The expected value of a normal UV* $\xi : N(e, \sigma)$ *is* $E[\xi] = e$.

Theorem 4.1 *Let* $\tilde{\eta}_i : N(a_i, \sigma_i)$, *with* $i = 1,2,\ldots,n$ *and* $\tilde{\lambda} : N(b, \sigma)$ *be independent normal UVs. Let* V_i *with* $i = 1,2,\ldots,n,$ *be nonnegative variables. Then for every* $\delta \in (0,1)$ *, the expression*

$$m\left(\sum_{i=1}^{n} \tilde{\eta}_i V_i \le \tilde{\lambda}\right) \ge \delta$$

is equivalent to

$$\sum_{i=1}^{n}\left(a_i + \frac{\sigma_i\sqrt{3}}{\pi} ln\left(\frac{\delta}{1-\delta}\right)\right)V_i \le b - \frac{\sigma\sqrt{3}}{\pi} ln\left(\frac{1-\delta}{\delta}\right).$$

By Theorem 4.1, the chance constraints in the equation (3.6) transform into a deterministic equivalent, that is, for all $j = 1,2,\ldots,m$,

$$m\left[\sum_{k=1}^{n}\tilde{\xi}_{jk}\psi_{jk}(x)\le\tilde{\lambda}_j\right]\ge\delta$$

is equivalent to

$$\sum_{k=1}^{n}\left(\xi_{jk}+\frac{\sigma_{jk}\sqrt{3}}{\pi}\ln\left(\frac{\delta}{1-\delta}\right)\right)\psi_{jk}(x)\le\lambda_j-\frac{\sigma_j\sqrt{3}}{\pi}\ln\left(\frac{1-\delta}{\delta}\right).$$

Also, Lemma 4.1 transforms the objective of the problem (3.6) into a precise value as,

$$E\left[\sum_{k=1}^{n}\tilde{\xi}_k\psi_k(x)\right]=\sum_{k=1}^{n}E\left[\tilde{\xi}_k\right]\psi_k(x)=\sum_{k=1}^{n}\xi_k\psi_k(x).$$

Consequently, when the coeffients are UVS that follow normal distribution, the model in equation (3.6) is transformed into the following crispy model,

$$\begin{aligned}&\min\sum_{k=1}^{n}\xi_k\psi_k(x)\\&\text{s.t.}\quad\sum_{k=1}^{n}\left(\xi_{jk}+\frac{\sigma_{jk}\sqrt{3}}{\pi}\ln\left(\frac{\delta}{1-\delta}\right)\right)\psi_{jk}(x)\le\lambda_j-\frac{\sigma_j\sqrt{3}}{\pi}\ln\left(\frac{1-\delta}{\delta}\right),\quad j=1,2,\ldots,m.\end{aligned}\tag{4.1}$$

4.2 Uncertain Optimization with Linear Uncertainty Distributions

Let the coefficients $\tilde{\xi}_k,\tilde{\xi}_{jk},\tilde{\lambda}_j$ in equation (3.6) be independent positive linear UVs, that is, $\tilde{\xi}_k:L(\xi_k^a,\xi_k^b),\tilde{\xi}_{jk}:L(\xi_{jk}^a,\xi_{jk}^b)$ and $\tilde{\lambda}_j:L(\lambda_j^a,\lambda_j^b)$, where $\xi_k^a,\xi_k^b,\xi_{jk}^a,\xi_{jk}^b,\lambda_j^a,\lambda_j^b$ are all positive real numbers with $0<\xi_k^a<\xi_k^b, 0<\xi_{jk}^a<\xi_{jk}^b, 0<\lambda_j^a<\lambda_j^b$.

We use Lemma 4.2 and Theorem 4.2 below to obtain a crisp equivalent of the model in equation (3.6) with linear UDs.

Theorem 4.2 *Let* $\tilde{\eta}_i(i=1,2,\ldots,n)$ *and* $\tilde{\lambda}$ *be independent linear UVs. That is,* $\tilde{\eta}_i:L(a_i,b_i)$ *with* $a_i<b_i$, *and* $\tilde{\lambda}:L(a_\lambda,b_\lambda)$ *with* $a_\lambda<b_\lambda$. *Let* $V_i(i=1,2,\ldots,n)$, *be nonnegative variables. Then for every* $\delta\in(0,1)$,

$$m\left(\sum_{i=1}^{n}\tilde{\eta}_iV_i\le\tilde{\lambda}\right)\ge\delta$$

is equivalent to

$$\sum_{i=1}^{n}((1-\delta)a_i+\delta b_i)V_i\le\delta a_\lambda+(1-\delta)b_\lambda.$$

Lemma 4.2 *([19]) The expected value of the linear UV* $\xi:L(a,b)$ *is* $E[\xi]=\frac{a+b}{2}$.

The chance constraints in equation (3.6) admit the following deterministic equivalent by Theorem 4.2, that is, for all $j=1,2,\ldots,m$,

$$m\left[\sum_{k=1}^{n}\tilde{\xi}_{jk}\psi_{jk}(x)\le\tilde{\lambda}_j\right]\ge\delta$$

is equivalent to

$$\sum_{k=1}^{n}\left((1-\delta)\xi_{jk}^a+\delta\xi_{jk}^b\right)\psi_k(x)\le\delta\lambda_j^a+(1-\delta)\lambda_j^b.$$

Again, by Lemma 4.2, the objective in the proposed problem (3.6) transforms into a deterministic one of the following form,

$$E\left[\sum_{k=1}^{n}\tilde{\xi}_k \psi_k(x)\right] = \sum_{k=1}^{n} E\left[\tilde{\xi}_k\right]\psi_k(x) = \sum_{k=1}^{n}\left(\frac{\xi_k^a + \xi_k^b}{2}\right)\psi_k(x).$$

The model in Eq. (3.6) is therefore equivalent to the following crispy model when the coefficients are UVs with a linear distribution,

$$\min \quad \sum_{k=1}^{n}\left(\frac{\xi_k^a + \xi_k^b}{2}\right)\psi_k(x)$$

$$\text{s.t.} \quad \sum_{k=1}^{n}\left((1-\delta)\xi_{jk}^a + \delta\xi_{jk}^b\right)\psi_{jk}(x) \le \delta\lambda_j^a - (1-\delta)\lambda_j^b, \quad j = 1,2,\ldots,m. \qquad (4.2)$$

5 NUMERICAL EXAMPLES

In this section, we demonstrate the effectiveness of the proposed model for the uncertain optimization program with numerical examples. In the situations of both linear and normal uncertain distributions, we solve the problem at some different predetermined degrees of uncertainty, using the same test problem.

Example 5.1 *Let us consider the following uncertain optimization problem.*

$$\begin{aligned} &\min \ f(x): \tilde{\xi}_1 x_1 + \tilde{\xi}_2 x_2 \\ &s.t \quad h_1(x): \tilde{\xi}_{11} x_1 + \tilde{\xi}_{12} x_2 = 10 \\ &\qquad h_2(x): \tilde{\xi}_{21} x_1 + \tilde{\xi}_{22} x_2 \le 5 \quad x_1 \ge 0, x_2 \ge 0. \end{aligned} \qquad (5.1)$$

As use the linear and normal uncertainty distributions for this problem.

Linear Uncertainty Distributions: Let the UVs in the model equation (5.1) follow linear UDs such that, $\tilde{\xi}_1 : L(10,20), \tilde{\xi}_2 : L(15,25), \tilde{\xi}_{11} : L(25,35), \tilde{\xi}_{12} : L(30,40), \tilde{\xi}_{21} : L(\frac{1}{3},\frac{2}{3}), \tilde{\xi}_{22} : L(\frac{2}{3},\frac{4}{3}).$

The use of equation (4.2) transforms the above model into the following deterministic optimization problem with specified values of δ .

$$\begin{aligned} &\min \ f(x): 15x_1 + 20x_2 \\ &\text{s.t.} \ h_1(x): (10\delta + 25)x_1 + (10\delta + 30)x_2 = 10 \\ &\qquad h_2(x): \frac{1}{3}(\delta+1)x_1 + \frac{2}{3}(\delta+1)x_2 \le 5 \end{aligned} \qquad (5.2)$$

We evaluate the problem at three distinct uncertainty levels(δ) and solve for the same.

Case 1: $\delta = \frac{1}{4}$. The problem in equation (5.2) becomes the following optimization problem,

$$\begin{aligned} &\min 15x_1 + 20x_2 \\ &\text{s.t.}\ 27.5x_1 + 32.5x_2 = 10 \\ &\frac{5}{12}x_1 + \frac{5}{6}x_2 \le 5 \end{aligned} \qquad (5.3)$$

In this case, the optimal solution is $x_1 = 0.364,\ x_2 = 0$ and the optimal objective value is $f_{min} = 5.455$.

Case 2: $\delta = \frac{1}{2}$. The problem in equation (5.2) transforms into the subsequent optimization problem,

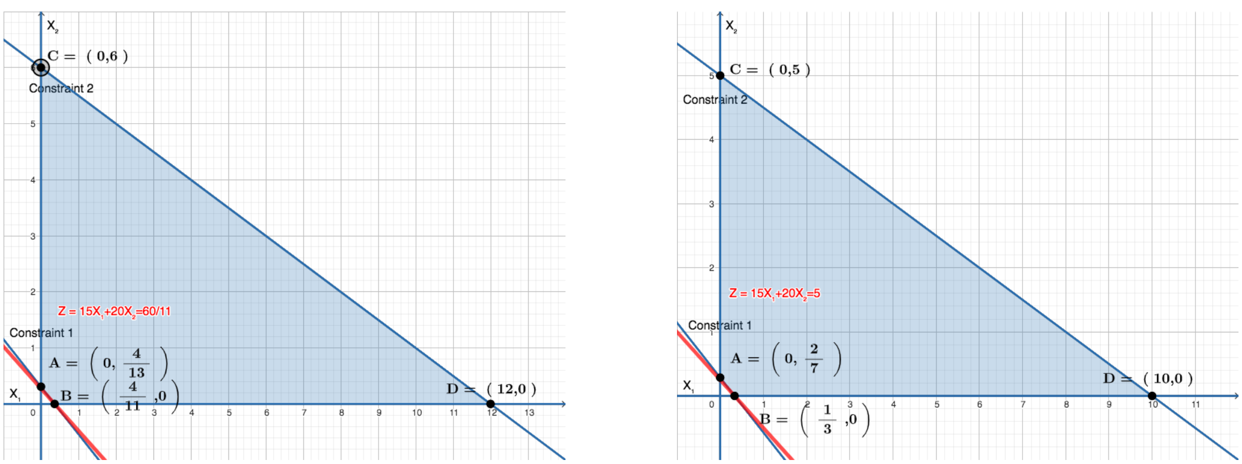
$$\min 15x_1 + 20x_2$$
$$s.t. \quad 30x_1 + 35x_2 \geq 10$$
$$\frac{1}{2}x_1 + x_2 \leq 5 \tag{5.4}$$

The problem in Eq. (5.4) gives the optimal solution is $x_1 = 0.333,\ x_2 = 0$ with optimal objective value $f_{min} = 5$.

Case 3: $\delta = \frac{3}{4}$. The problem in Eq. (5.4) becomes the following optimization problem,

$$\min 15x_1 + 20x_2$$
$$s.t. \quad 32.5x_1 + 37.5x_2 = 10$$
$$\frac{7}{12}x_1 + \frac{7}{6}x_2 \leq 5 \tag{5.5}$$

In this case, the problem in Eq. (5.4) admits the optimal solution $x_1 = 0.3077,\ x_2 = 0$ with optimal objective value $f_{min} = 4.6154$.



(a) $\delta = \frac{1}{4}$ (b) $\delta = \frac{1}{2}$

Figure 2: Constrained region for $\delta = \frac{1}{4}$ and $\delta = \frac{1}{2}$ under linear UVs.

Table 1: Optimal solutions with linear UD

Uncertanty level (δ)	Optimal solution(x_1, x_2)	Objective value(f)
0.1	(0.385, 0)	5.769
0.2	(0.370, 0)	5.555
0.3	(0.357, 0)	5.357
0.4	(0.349, 0)	5.172
0.5	(0.333, 0)	5.000
0.6	(0.323, 0)	4.839

0.7	(0.313, 0)	4.687
0.8	(0.303, 0)	4.545
0.9	(0.294, 0)	4.412

Table 1 shows the optimal solutions obtained for problem (5.2) under linear UVs along with their corresponding objective values. It is observed from the table that the optimal objective value decreases as δ increases.

Normal Uncertainty Distributions: Let the UV's in Eq. (5.1) follow the normal UDs as, $\tilde{\xi}_1 : N(8,2), \tilde{\xi}_2 : N(10,3), \tilde{\xi}_{11} : N(15,4), \tilde{\xi}_{12} : N(20,5), \tilde{\xi}_{21} : N(\frac{1}{2},\frac{1}{10}), \tilde{\xi}_{22} : N(\frac{4}{3},\frac{1}{5}).$ We use equation(??) and the normal UDs defined above to transform the model in equation(??) into a deterministic optimization problem as below,

$$\begin{aligned} &\min \quad f(x): 8x_1 + 10x_2 \\ &\text{s.t.} \quad h_1(x): \left(15 + \frac{4\sqrt{3}}{\pi}\ln\left(\frac{\delta}{1-\delta}\right)\right)x_1 + \left(20 + \frac{5\sqrt{3}}{\pi}\ln\left(\frac{\delta}{1-\delta}\right)\right)x_2 = 10 \\ &\qquad h_2(x): \left(\frac{1}{2} + \frac{\sqrt{3}}{10\pi}\ln\left(\frac{\delta}{1-\delta}\right)\right)x_1 + \left(\frac{4}{3} + \frac{\sqrt{3}}{5\pi}\ln\left(\frac{\delta}{1-\delta}\right)\right)x_2 \le 5 \end{aligned} \tag{5.6}$$

We present the solution methodology for the above problem at three distint values of δ and study the optimal solutions.

Case 1: $\alpha = \frac{1}{4}$. For this value of δ, the problem in Eq. (5.6) transforms into the subsequent optimization problem,

$$\begin{gathered} \min 8x_1 + 10x_2 \\ \text{s.t.} 12.577\, x_1 + 16.971\, x_2 = 10 \\ 0.439\, x_1 + 1.212\, x_2 \le 5 \end{gathered} \tag{5.7}$$

In this case, the optimal solution is $x_1 = 0.795,\ x_2 = 0$ and the optimal objective value is $f_{min} = 6.36$.

Case 2: $\alpha = \frac{1}{2}$. In this case, the transformed optimization problem of Eq. (5.6) is given as,

$$\begin{gathered} \min 8x_1 + 10x_2 \\ s.t. 15x_1 + 20x_2 = 10 \\ \frac{1}{2}x_1 + \frac{4}{3}x_2 \le 5 \end{gathered} \tag{5.8}$$

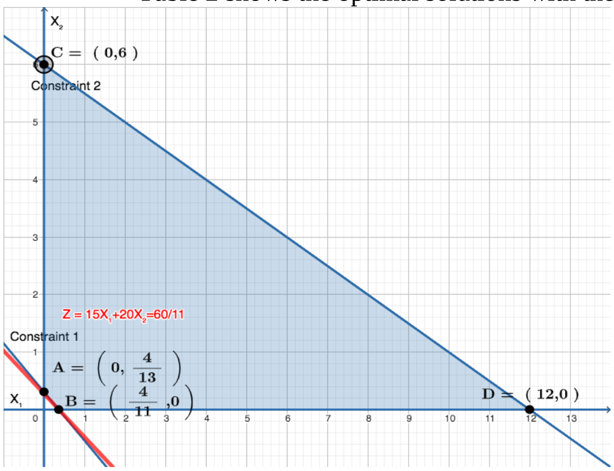
In this case, the optimal solution is $x_1 = 0.667,\ x_2 = 0$ and the optimal objective value is $f_{min} = 5.333$.

Case 3: $\alpha = \frac{3}{4}$. For this value of δ, the problem in Eq. (5.9) transforms into the subsequent optimization problem,

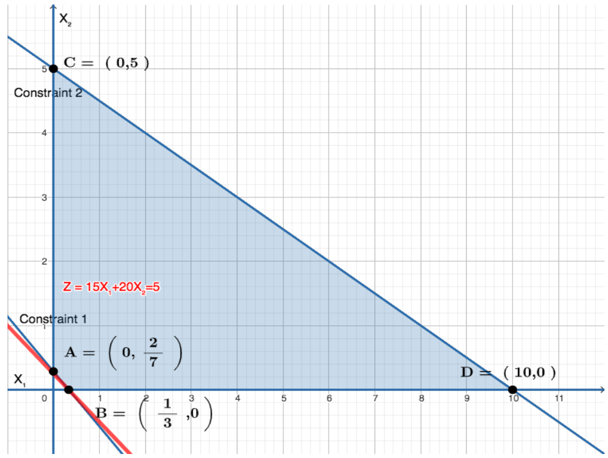
$$\begin{gathered} \min 8x_1 + 10x_2 \\ s.t. 17.423\, x_1 + 23.028\, x_2 = 10 \\ 0.560\, x_1 + 1.454\, x_2 \le 5 \end{gathered} \tag{5.9}$$

The optimal solution is $x_1 = 0.574$, $x_2 = 0$ and the optimal objective value is $f_{min} = 4.592$.

Table 2 shows the optimal solutions with the assignined weights into the weighted sum function.



(a) $\delta = \frac{1}{4}$



(b) $\delta = \frac{1}{2}$

Figure 3: Constrained region for $\delta = \frac{1}{4}$ and $\delta = \frac{1}{2}$ under normal UVs.

Table 2: Optimal solutions with normal UD

Uncertanty level (δ)	Optimal solution(x_1, x_2)	Objective value(f)
0.1	(0.985, 0)	7.878
0.2	(0.837, 0)	6.699
0.3	(0.762, 0)	6.093
0.4	(0.709, 0)	5.671
0.5	(0.667, 0)	5.333
0.6	(0.629, 0)	5.033
0.7	(0.593, 0)	4.743
0.8	(0.554, 0)	4.430
0.9	(0.504, 0)	4.031

The optimal solutions obtained for the problem in Eq. (5.2) under linear UV is shown in Table 2, with their corresponding objective values. It is observed from the table that the optimal objective values decreases as δ increases.

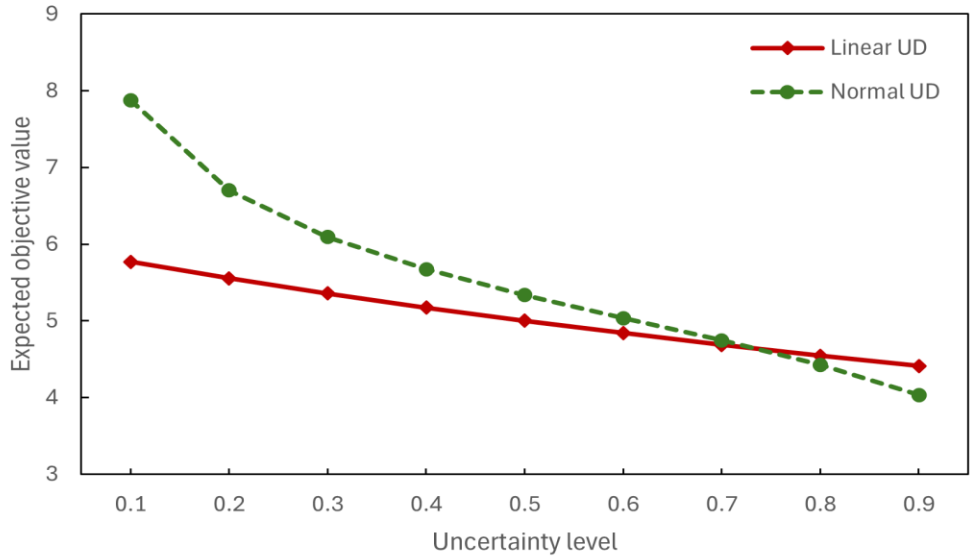
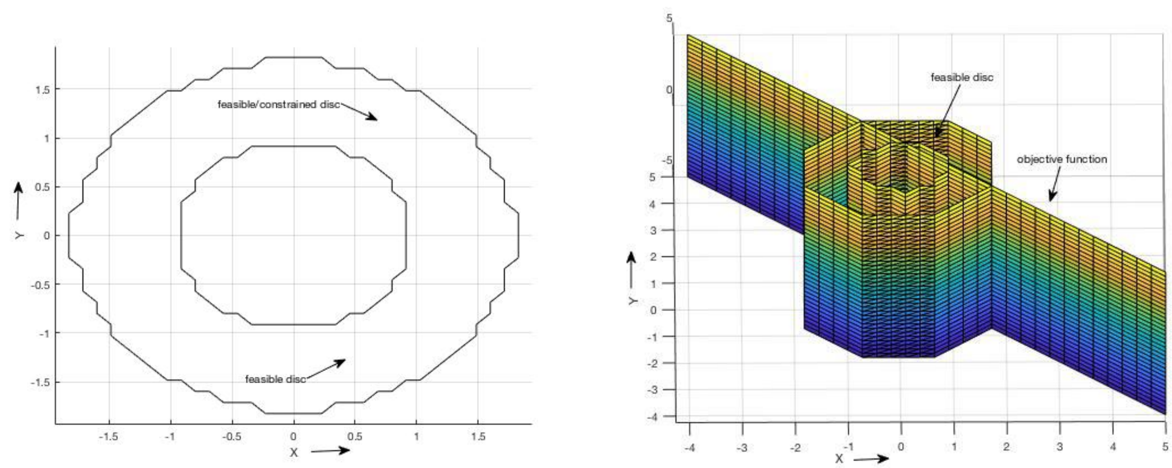


Figure 4: Comparison of the optimal objective values for linear and normal UVs.

Example 5.2 *In this example we consider a more general scenario of uncertain optimization problem where the constraints are nonlinear functions and both the coefficients and the lmits of the constraints are assumed to be UVs.*

$$\begin{aligned} \max\ & f(x) = \tilde{\xi}_1 x_1 + \tilde{\xi}_2 x_2 \\ \text{s.t.}\ & h_1(x) = \tilde{\xi}_{11} x_1^2 + \tilde{\xi}_{12} x_2^2 \geq \tilde{\lambda}_1 \\ & h_2(x) = \tilde{\xi}_{21} x_1^2 + \tilde{\xi}_{22} x_2^2 \leq \tilde{\lambda}_2 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned} \tag{5.10}$$

The above optimization problem is maximization problem to be solved in a circular disc basically and moreover, as in example 5.1, we use the linear and normal uncertainty distributions for this problem.



2D view of feasible disc 3D view of feasible disc

Figure 5: Disc as feasible region for problem (5.10).

Linear Uncertainty Distributions: Let the UVs in the problem in Eq. (5.10) follow linear UDs such that $\tilde{\xi}_1 : L(5,7), \tilde{\xi}_2 : N(\frac{9}{2},\frac{5}{2}), \tilde{\xi}_{11} : L(4,5), \tilde{\xi}_{12} : L(3,\frac{5}{2}), \tilde{\lambda}_1 : L(1,2), \tilde{\xi}_{21} : L(\frac{5}{2},\frac{7}{2}), \tilde{\xi}_{22} : L(4,6), \tilde{\lambda}_2 : N(3,4).$

Use of linear UDs transform the model in Eq. (5.10) into a deterministic optimization problem as,

$$\begin{aligned} &\max\ 6x_1 + 5x_2 \\ &\text{s.t.}\ (5-\delta)x_1^2 + \frac{1}{2}(9-3\delta)x_2^2 \geq 2-\delta \\ &(\frac{5}{2}+\delta)x_1^2 + (4+2\delta)x_2^2 \leq 4-\delta. \end{aligned} \quad (5.11)$$

We evaluate the solution at three distinct values of δ and show the optimal result for more in Table 5.2.

Case 1: $\delta = \frac{1}{4}$. The problem in Eq. (5.11) becomes the following optimization problem,

$$\begin{aligned} &\max\ 6x_1 + 5x_2 \\ &\text{s.t.}\ 4.75x_1^2 + 4.13x_2^2 \geq 1.75 \\ &2.75x_1^2 + 4.5x_2^2 \leq 3.75 \end{aligned} \quad (5.12)$$

The optimal solution in this case is $x_1 = 0.978,\ x_2 = 0.498$ and the corresponding optimal objective value is $f_{max} = 8.362$.

Case 2: $\delta = \frac{1}{2}$. The transformed problem is given as

$$\begin{aligned} &\max\quad 6x_1 + 5x_2 \\ &\text{s.t.}\ 4.5x_1^2 + 3.75x_2^2 \geq 1.5 \\ &3x_1^2 + 5x_2^2 \leq 3.5 \end{aligned} \quad (5.13)$$

The optimal solution is $x_1 = 0.907,\ x_2 = 0.454$ with optimal objective value $f_{max} = 7.71$.

Case 3: $\delta = \frac{3}{4}$. The transformed optimization problem is,

$$\begin{aligned} &\max\ 6x_1 + 5x_2 \\ &\text{s.t.}\ 4.25x_1^2 + 3.37x_2^2 \geq 1.25 \\ &3.25x_1^2 + 5.5x_2^2 \leq 3.25 \end{aligned} \quad (5.14)$$

The above problem admits the optimal solution is $x_1 = 0.842,\ x_2 = 0.415$ with optimal objective value $f_{max} = 7.126$.

Table 3: Optimal solutions for linear UD

Uncertanty level (δ)	Optimal solution(x_1, x_2)	Objective value(f)
0.1	(1.024, 0.528)	8.728
0.2	(0.993, 0.508)	8.500
0.3	(0.964, 0.489)	8.227
0.4	(0.935, 0.470)	7.965
0.5	(0.907, 0.454)	7.714
0.6	(0.880, 0.437)	7.472

0.7	(0.855, 0.422)	7.239
0.8	(0.829, 0.407)	7.014
0.9	(0.805, 0.393)	6.796

Table 3 shows the optimal solutions obtained for the problem in Eq.(5.10) under linear UVs along with their corresponding optimal objective values. Although it is a maximization problem, we can still see from the table that the optimal objective value decrease as δ increases.

Normal Uncertainty Distributions: Let the UVs in Eq. (5.10) follow normal UDs respectively as, $\tilde{\xi}_1 : N(5,2), \tilde{\xi}_2 : N(4,2), \tilde{\xi}_{11} : N(2,\frac{1}{2}), \tilde{\xi}_{12} : N(3,\frac{1}{5}), \tilde{\lambda}_1 : N(2,\frac{1}{10}), \tilde{\xi}_{21} : N(4,\frac{1}{2}), \tilde{\xi}_{22} : N(6,3), \tilde{\lambda}_2 : N(5,1).$

We use Eq. (4.1) and the normal UDs to transform the model (5.10) into a deterministic optimization problem as,

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 \\ \text{s.t.} \quad & \left(2 - \frac{\sqrt{3}}{2\pi}\ln\left(\frac{1-\delta}{\delta}\right)\right)x_1^2 + \left(3 - \frac{\sqrt{3}}{5\pi}\ln\left(\frac{1-\delta}{\delta}\right)\right)x_2^2 \geq \left(2 - \frac{\sqrt{3}}{10\pi}\ln\left(\frac{1-\delta}{\delta}\right)\right) \\ & \left(4 + \frac{\sqrt{3}}{2\pi}\ln\left(\frac{\delta}{1-\delta}\right)\right)x_1^2 + \left(6 + \frac{3\sqrt{3}}{\pi}\ln\left(\frac{\delta}{1-\delta}\right)\right)x_2^2 \leq \left(5 - \frac{\sqrt{3}}{\pi}\ln\left(\frac{1-\delta}{\delta}\right)\right) \end{aligned} \quad (5.15)$$

Case 1: $\delta = \frac{1}{4}$. The problem in Eq. (5.15) transforms into the following optimization problem,

$$\begin{aligned} & \max 5x_1 + 4x_2 \\ & \text{s.t.}\, 1.697\,x_1^2 + 2.878\,x_2^2 \geq 1.939 \\ & 3.697\,x_1^2 + 4.182\,x_2^2 \leq 4.394. \end{aligned} \quad (5.16)$$

The optimal solution is $x_1 = 0.871,\ x_2 = 0.616$ with optimal objective value $f_{max} = 6.820$.

Case 2: $\alpha = \frac{1}{2}$. In this case, the transformed problem of Eq. (5.11) is given as,

$$\begin{aligned} & \max 5x_1 + 4x_2 \\ & \text{s.t.}\, 2x_1^2 + 3x_2^2 \geq 2 \\ & 4x_1^2 + 6x_2^2 \leq 5. \end{aligned} \quad (5.17)$$

The optimal solution in this case is $x_1 = 0.936,\ x_2 = 0.499$ and $f_{max} = 6.677$.

Case 3: $\alpha = \frac{3}{4}$. The problem in equation (??) transforms into the optimization problem as,

$$\begin{aligned} & \max 5x_1 + 4x_2 \\ & \text{s.t.}\, 2.302\,x_1^2 + 7.817\,x_2^2 \geq 2.060 \\ & 4.302\,x_1^2 + 7.817\,x_2^2 \leq 5.605. \end{aligned} \quad (5.18)$$

The optimal solution is $x_1 = 0.981,\ x_2 = 0.432$ and $f_{max} = 6.637$.

Table 4 shows the optimal solutions along with corresponding objective values. Similarly, in this case, the optimal objective values decrease as δ increases.

Table 4: Optimal solutions for normal UD

Uncertanty level (δ)	Optimal solution(x_1, x_2)	Objective value(f)
0.1	(0.763, 0.875)	7.317
0.2	(0.849, 0.663)	6.896
0.3	(0.889, 0.582)	6.771
0.4	(0.915, 0.534)	6.711
0.5	(0.936, 0.499)	6.677
0.6	(0.954, 0.470)	6.655
0.7	(0.972, 0.444)	6.641
0.8	(0.991, 0.412)	6.634
0.9	(1.016, 0.389)	6.630

Figure 6: Comparison of the optimal objective values for linear and normal Uvs.

Discussion: In both examples only linear and normal UVs are assumed, and it is obvious to observe from the data tables that in each case, as the uncertainty level increases, the optimal objective values decreses. Fig.5 and Fig.6 make it somewhat easier to support the validation of this statement. Moreover, in this two figures, a comparison of the optimal objective values are plotted against the change of uncertainty levels and they both have the same characteristics.

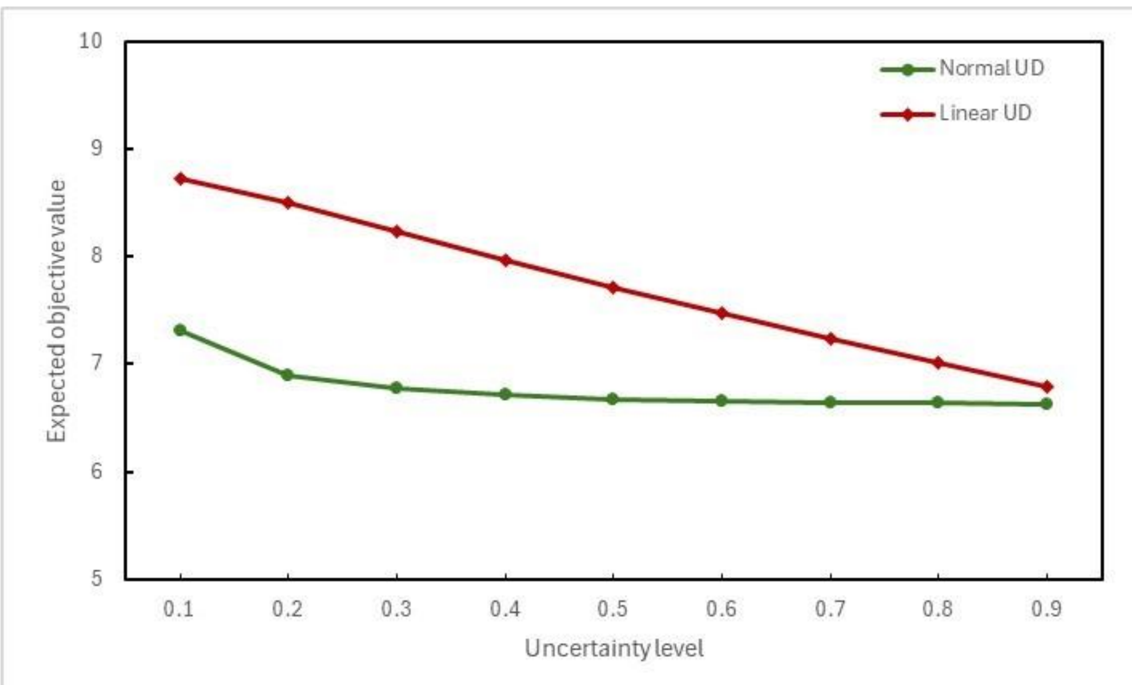


Figure 6: Comparison of the optimal objective values for linear and normal UVs.

6 Conclusion

In various disciplines, there has been a growing interest in solving uncertain optimization problems. This topic has parallelly been explored by researchers in both mathematical optimization and decision making under deep uncertainty. The former focuses mostly on theory developments, while the later concentrates on practical problems. In traditional optimization problems the parameters are deterministic and precise. However, in real-world optimization problems, the situaion is not the same, and parameters are often uncertain and imprecise. Since the recent past, uncertainty theory has been used to address these types of uncertain problems. In this paper, we solve an uncertain single objective optimization problem under linear and normal uncertainty distribution. We develop the equivalent chance-constrained models and solve the problem by transforming it in into conventional optimization problem with crisp coefficinets. To illustrate the effectiveness of the methods and algorithms, two numerical examples are provided in this context. We solve two numerical problems under linear and normal uncertainty distributions for different uncertainty levels and obtain the corresponding expected values of the objective function. The expected cost function is then plotted against the uncertainty level and the graph shows the comparison downfall of the objective values as the uncertainty level increases.

In this paper, our main focus was to solve an uncertain single objective optimization problem when the coefficients are assumed to be UVs that follow linear and normal UDs and show how the expected values of the objective function change in relation to uncertainty levels. We are in a satisfactory comment that we are successful in doing the same.

Declairation: The data that support the findings of this study are generated by the authors and are available from the corresponding author upon request.

Also, the author confirms that the data generated in this manuscript are not included in any published paper or in any manuscript that is under consideration for review. Finally, the authors have no conflict of interest.

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