# Discretizing Continuous Distributions - A Comparative Study

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**Abstract:** *Recently discrete distributions have played a significant role in modeling real world scenarios. Though a large variety of discrete distributions are originated, the existing distributions are unfit to model many practical situations. Now a days, various discretization methods are proposed to derive discrete versions of continuous distributions especially for modeling survival data. In this article, we consider a review of various discretization methods and the distributions thus discretized so far. The discrete analogues of some continuous distributions viz. Burr, Exponential, Gamma, Generalized Exponential, Laplace, Log Cauchy, Normal, Pareto, Rayleigh and Skew Laplace are reviewed. A comparison on various discretization methods is also carried out.*

**Keywords:** *Discretization, Generalized Exponential Distribution, Hazard rate function, Laplace Distribution, Rayleigh Distribution.*

## 1. Introduction

In the present scenario, discrete distributions are widely used in modeling survival data instead of continuous distributions. Though a lot of continuous lifetime distributions are proposed by many researchers, it is not always possible to measure the life time of a gadget on a continuous scale. When we use a discrete distribution in modeling lifetime data, it usually leads to a multinomial distribution. But there exists many practical situations which urged to more lifetime distributions to model and hence many continuous distributions have to be discretized. There are a lot of research work s related to this are available in literature viz. Lisman and van Zuylen (1972), Kemp(1997), Das gupta (1993) and Szablowski (2001), Roy (2003,2004), Inusahand Kozubowski (2006), Kozubowski and Inusah (2006), Krishna and Pundir(2009), Alzaatrech et al.(2012), Nekoukhou et al. (2012), Nekoukhou and Bidram (2015), Lekshmi and Sebastian (2014), Chakraborty and Chakravarthy(2016), Abebe and Shanker (2018), Krishnakumari and Dais (2020), Krishnakumari and Dais (2021).

The remaining part of this article is organized as follows. The discretization methods proposed by many researchers are reviewed in Section 2. In Section 3, discrete analogues of continuous life time distributions derived using the methods proposed in Section 2 are reviewed. A comparison of various discretization methods is discussed in Section 4.

## 2. Discretization Methods

Discrete distributions play an extensive role in modeling real life situations. Hence a large number of discrete distributions are proposed and studied by many researchers. For details, see the books by Balakrishnan and Nevzorov (2003), Jonson et al. (2005) and Consul and Famoye (2006). In this section, we made a survey on the discretization methods proposed by many researchers.

### 2.1 Method 1

A discrete analogue of Pearson's continuous system is developed by Katz (1945) using the relationship

$$\frac{P_x+1}{P_x}=\frac{a+bx}{1+x}; \quad x=0,1,2\dots \tag{1}$$

Kemp (1968) generated a family of existing discrete distributions by generalizing (1) discussed in Jonson et al. (2005).

### 2.2 Method 2

For any continuous random variable on R, with probability density function f(x), its discrete probability mass function is given by

$$P(X=x)=\frac{f(x)}{\sum_{u=-\infty}^{+\infty} f(u)}, x=\cdots,-2,-1,0,1,2\dots \tag{2}$$

**2.3 Method 3**

Another method is from reliability perspective proposed by Roy (2003, 2004). If the survival function of a continuous random variable is denoted by $S(x) = P(X \geq x)$, and if times are grouped into unit intervals, the discrete observed variable $dX = [X]$, the largest integer less than or equal to X, has the probability function

$$\boldsymbol{P(dX = x) = S(x) - S(x+1); \quad x = 0, 1, 2 \ldots.} \tag{3}$$

**2.4 Method 4**

If X is a continuous random variable belongs to the extended exponential family with distribution function

$$\mathbf{F(x) = 1 - e^{[-\propto k_\theta(x)\,]}},$$

Then its discrete version belongs to the telescopic family of distributions defined by Roknabadi et al. (2009), which has the probability mass function

$$\mathbf{P(X = x) = q^{k_\theta(x)} - q^{k_\theta(x+1)}; \quad x = 0, 1, 2 \ldots} \tag{4}$$

Where $k_\theta(x)$ is strictly increasing function of x, $0<q<1$.

**2.5 Method 5**

Ganji and Gharari (2018) presented a new method for discretization of most of continuous distributions, where their probability density functions consists of the monomial Taylor and exponential function. They use discrete fractional calculus for showing the existence of delta and nabla distributions and then apply time scales for definition of delta and nabla discrete distributions.

**2.6 Method 6**

To meet the need of fitting discrete-time reliability and survival datasets, Yari and Tondpour (2018) proposed this new method which provides three two-stage composite discretization methods. All of these three methods consist of two stages where in the first stage a new continuous random variable is constructed from the underlying continuous random variable and in the second stage, a discrete analogue of this new continuous random variable is derived by maintaining the same hazard rate function. The three methodologies used are

**2.6.1 Methodology I**

Here, in the first stage a continuous random variable X with cumulative distribution function F(x) and support $[0, \infty)$ is used to construct a new continuous random variable $X_1$ having hazard rate function

$$\mathbf{h_{X1}(x) = e^{-F(x)}, x \geq 0} \tag{5}$$

In the second stage, a discrete analogue Y of $X_1$ is derived by using the following methodology where hazard rate function of Y retains the form of hazard rate function of $X_1$. If the continuous random variable $X_1$ has survival function $S_{X1}$ $(x)$ and hazard rate function $h_{X1}$ $(x)$ then the survival function of the discrete analogue Y is given by

$$\boldsymbol{S_Y(k) = \big(1 - h_{X1}(1)\big)\big(1 - h_{X1}(2)\big) \ldots \big(1 - h_{X1}(k-1)\big); k = 1, 2, \ldots m.}$$

The corresponding probability mass function is therefore

$$\mathbf{P(Y = k)} = \begin{cases} \mathbf{h_{X1}(0)); \quad k = 0} \\ \mathbf{((1 - h_{X1}(1))\left((1 - h_{X1}(2)) \ldots (1 - h_{X1}(k-1))\right) h_{X1}(k);} \quad k = 1,2,..\mathbf{m} \\ \mathbf{0; \quad otherwise.} \end{cases} \tag{6}$$

If P(Y=k) is such that the total probability is not equal to one, then we shall multiply every P(y) by the positive constant w that will ensure the total probability equals to one. Hence the probability mass function takes the form

$$P(Y=y) = \begin{cases} w; & y=0 \\ w h_{X_1}(y) \prod_{i=1}^{y-1}(1-h_{X_1}(i)); & y=1,2,\dots m \\ 0; & \text{otherwise} \end{cases} \tag{7}$$

Where m can be finite or infinite since hX1(x) is always between zero and one. Now, by using (7) the resulting pmf of Y in new methodology is

$$P(Y=y) \begin{cases} w; & y=0 \\ w e^{-F_X(x)} \prod_{i=1}^{y-1}(1-e^{-F_X(i)}); & y=1,2,\dots m \\ 0; & \text{otherwise} \end{cases} \tag{8}$$

**2.6.2 Methodology II**

In this method, in the first stage a new continuous random variable $X_1$ having hazard rate function

$$h_{X1}(x) = \frac{2F_X(x)}{1+F_X(x)}$$

By using continuous random variable X with cumulative distribution function $F_X(x)$ and support [0, ∞) is constructed. Then in the second stage, a discrete analogue Y of $X_1$ is derived by using (7). Also note that discrete distributions obtained in this methodology has increasing hazard rate function.

**2.6.3 Methodology III**

Here also in the first stage a continuous random variable X with cumulative distribution function $F_X(x)$ and Support [0, ∞) is used to construct a new continuous random variable $X_1$ having hazard rate function

$$h_{X1}(x) = \frac{1}{f_X(x)+1} \qquad x \geq 0.$$

Then in the second stage, a discrete analogue Y of $X_1$ is derived by using (7). Here the hazard rate function of Y is increasing (decreasing) on (*a*,*b*) where $a, b \in R^+$if and only if $f_X(x)$is decreasing (increasing) on same interval.

In the first two methods, hazard rate functions of discrete analogues are decreasing and increasing respectively and in the third method they can be in-creasing, U shaped or modified unimodal. An important advantage of the method is that discrete analogues obtained have monotonic and non-monotonic hazard rate functions.

## 3. Discrete Analogue of Continuous Distributions

Usually a continuous distribution is designated by its probability density function, distribution function, moments, hazard rate function etc. A discrete analogue of continuous distribution is derived by maintaining one or more discriminative property of the continuous distribution. In this section, various discretized distributions derived through the various discretization methods are reviewed.

### 3.1 Discretized distributions using Method 2

Here we consider the discrete analogue of normal, Laplace, skew Laplace and generalized exponential distributions.

#### 3.1.1 Discrete Normal Distribution

The discrete Normal distribution was derived by Kemp (1997) from *N* ($\mu, \sigma$) by substituting

$$\boldsymbol{\lambda = e^{\frac{-(1-2\mu)}{2\sigma^2}} \text{ and } q = e^{\frac{-1}{\sigma^2}}}$$

The pmf is

$$P(X = x) = \frac{\lambda^x q^{\frac{x(x-1)}{2}}}{\sum_{j=-\infty}^{+\infty} \lambda^j q^{\frac{j(j-1)}{2}}}, x = \cdots, -2, -1, 0, 1, 2 \ldots \tag{9}$$

The plot of Discrete Normal distribution is given in Figure.1.

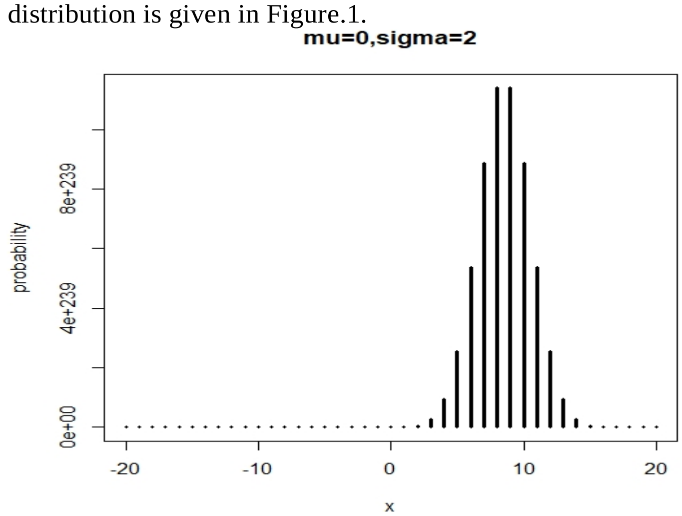


Figure.1.Discrete Normal distribution

This distribution is characterized by maximum entropy for specified mean and variance and integer support on ($-\infty, +\infty$).). It can be derived as the distribution of the difference of two related Heine variables.

#### 3.1.2 Discrete Laplace Distribution

Following Kemp (1997) who defined discrete Normal distribution, a discrete version of Laplace distribution was proposed and studied by Inusah and Koszubowski (2006).

Consider the classical Laplace distribution with probability density function

$$f(x) = \frac{1}{2\sigma} e^{\frac{-|x|}{\sigma}}; \quad x \in R, \sigma > 0 \tag{10}$$

And distribution function

$$F(x) = 1 - \frac{e^{\frac{x}{\sigma}}}{2}.$$

The survival function and failure rate are $\mathbf{S(x)} = \frac{\mathbf{e}^{\frac{-x}{\sigma}}}{2}$ and $\mathbf{r(x)} = \frac{1}{\sigma}$ respectively. Using method 2 in section 2, the discrete version of (10) is given by

$$\mathbf{f(x)=\frac{1-p}{1+p}p^{|x|};\quad x\ldots,-2,-1,0,1,2} \tag{11}$$

Where $\boldsymbol{p = e^{(\frac{-1}{\sigma})}}$.

This is the pmf of discrete Laplace distribution. Also this distribution contributes many properties of the classical Laplace distribution. The cumulative distribution function of discrete Laplace distribution is

$$\mathbf{F(x)} = \begin{cases} \dfrac{\boldsymbol{p}^{-[\boldsymbol{x}]}}{\mathbf{1}+\boldsymbol{p}}; & \mathbf{x} < 0 \\ \mathbf{1}-\dfrac{\boldsymbol{p}^{[\boldsymbol{x}]+\mathbf{1}}}{\mathbf{1}+\boldsymbol{p}}; & \boldsymbol{x} \geq \mathbf{0} \end{cases}$$

Where [.] is the greatest integer function.

Also the mean, variance, characteristic function, survival function and failure rate are obtained respectively as

$$\boldsymbol{Mean} = \mathbf{0}$$

$$\mathbf{V\,ariance} = \frac{\mathbf{2p}}{(\mathbf{1-p})^{2}}$$

$$\textbf{Characteristic function} = \frac{(\mathbf{1-p})^{2}}{(\mathbf{1-e^{it}p})(\mathbf{1-e^{it}p})}$$

$$\textbf{Survival function} = \frac{\mathbf{p}^{\{\mathbf{x}\}+\mathbf{1}}}{\mathbf{1+p}};\quad \mathbf{x \geq 0}$$

$$\textbf{Failurerate} = \frac{\mathbf{1-p}}{\mathbf{p}}$$

It is seen that expressions for the probability density function, distribution function, characteristic function, the mean and the variance are obtained in closed form. This discrete model is useful for evaluating the uncertainty in droughts, floods, El Ninos, spells etc. Its applications can be extended to civil engineering and insurance industry etc. For details see Inusah and Kozubowski (2006). The plot of Discrete Laplace distribution is given in Figure.2.

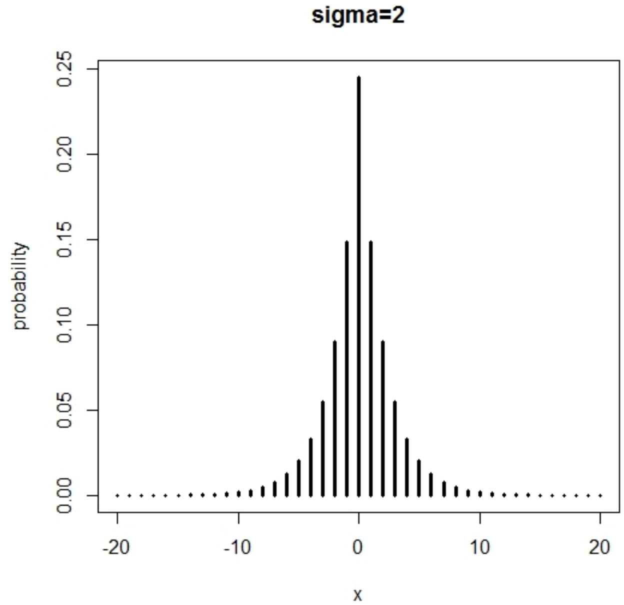


Figure.2. Discrete Laplace distribution

### 3.1.3 Skew Laplace Distribution on Integers

Kozubowski and Inusah (2006) proposed discrete version of skew Laplace distribution. The pdf of the skew Laplace distribution with a scale parameter $\sigma>0$ and the skewness parameter (see Kotz et al. (2001)) is given by

$$f(x)=\begin{cases} \dfrac{\mathbf{k}}{\boldsymbol{\sigma}(\mathbf{1}+\mathbf{k}^{\mathbf{2}})}\mathbf{e}^{\frac{\mathbf{x}}{\mathbf{k}\boldsymbol{\sigma}}}; & \mathbf{x}<0 \\ \dfrac{\mathbf{k}}{\boldsymbol{\sigma}(\mathbf{1}+\mathbf{k}^{\mathbf{2}})}\mathbf{e}^{\frac{-\mathbf{k}\mathbf{x}}{\boldsymbol{\sigma}}}; & \mathbf{x}\geq \mathbf{0} \end{cases} \tag{12}$$

Using method-2, its discrete version takes on an explicit form in terms of the parameters $\boldsymbol{p}=\boldsymbol{e}^{\frac{-k}{\sigma}}$ and $\boldsymbol{q}=\boldsymbol{e}^{\frac{-1}{k\sigma}}$, $p\in(0,1)$ and $q\in(0,1)$ as

$$f(x;p,q)=\begin{cases} \dfrac{(\mathbf{1}-\mathbf{p})(\mathbf{1}-\mathbf{q})}{\mathbf{1}-\mathbf{pq}}\mathbf{q}^{|\mathbf{x}|}; \mathbf{x}=\mathbf{0},-\mathbf{1},-\mathbf{2},-\mathbf{3},\ldots \\ \dfrac{(\mathbf{1}-\mathbf{p})(\mathbf{1}-\mathbf{q})}{\mathbf{1}-\mathbf{pq}}\mathbf{p}^{\mathbf{x}}; \mathbf{x}=\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\ldots \end{cases} \tag{13}$$

The distribution function is

$$F(x;p,q)=\begin{cases} \dfrac{(\mathbf{1}-\mathbf{p})(\mathbf{q}^{-|\mathbf{x}|})}{\mathbf{1}-\mathbf{pq}} & \mathbf{x}<0 \\ \mathbf{1}-\dfrac{(\mathbf{1}-\boldsymbol{q})(\boldsymbol{p}^{|\boldsymbol{x}|+\mathbf{1}})}{\mathbf{1}-\boldsymbol{pq}}; & \mathbf{x}\geq \mathbf{0} \end{cases}$$

While the characteristic function is

$$\phi(t, p, q) = \frac{(1-p)(1-q)}{(1-e^{it}p)(1-e^{-it}q)}$$

The plot of Skew Laplace distribution on integers is given in Figure.3.

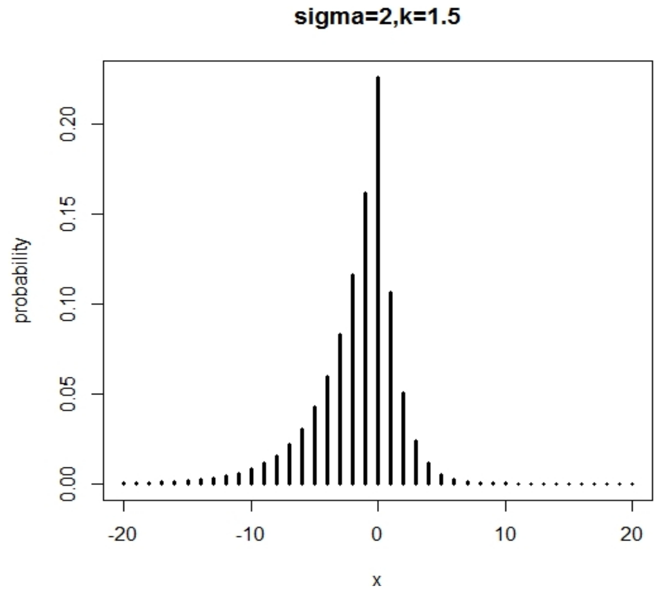


Figure.3 Skew Laplace distribution on integers

This distribution possesses many properties like infinite divisibility, unimodality, maximum entropy property and closure property with respect to geometric compounding of the skew Laplace distribution on the real line.

**3.1.4 A Skewed Generalized Discrete Laplace Distribution**

A generalized discrete Laplace distribution which appears as the difference of two independent negative binomial variables with common dispersion parameter was proposed by Lekshmi and Sebastian (2014). Its characteristic function has the form

$$\phi(t, p, q) = \left[\frac{(1-p)(1-q)}{(1-e^{it}p)(1-e^{-it}q)}\right]^{\beta} \quad (14)$$

When $\beta = 1$, it reduces to the characteristic function of discrete skew Laplace distribution. This distribution is suitable for modeling currency exchange rates.

**3.1.5 Discrete Generalized Exponential Distribution**

Consider the generalized exponential distribution of Gupta & Kundu (1999) having pdf.

$$f(x, \propto, \lambda) = \alpha\lambda[1 - e^{(-\lambda x)}]^{\alpha-1} e^{(-\lambda x)}; \;\; x > 0, \alpha > 0, \lambda > 0 \quad (15)$$

With cumulative distribution function

$$F(x, \alpha, \lambda) = [1 - e^{(-\lambda x)}]^{\alpha} \;\; x > 0$$

Survival function is

$$S(x) = 1 - [1 - e^{(-\lambda x)}]^{\alpha}$$

Its failure rate is

$$r(\mathbf{x}) = \frac{\boldsymbol{\alpha\lambda}[\mathbf{1} - \mathbf{e}^{(-\boldsymbol{\lambda}\mathbf{x})}]^{\boldsymbol{\alpha}-\mathbf{1}}\mathbf{e}^{(-\boldsymbol{\lambda}\mathbf{x})}}{\mathbf{1} - [\mathbf{1} - \mathbf{e}^{(-\boldsymbol{\lambda}\mathbf{x})}]^{\boldsymbol{\alpha}}}$$

This distribution has increasing and decreasing failure rate depending on the value of the shape parameter $\alpha$. When $\alpha$=1, failure rate is constant. The discrete version of generalized exponential distribution is proposed by Nekoukhou et al. (2012) and its pmf is

$$\boldsymbol{f(y,\alpha,p) = kp^{y-1}(1 - p^{y})^{\alpha-1}; 0} < p < 1, p = \boldsymbol{e^{-\lambda}}, \boldsymbol{\alpha} > 0, y \in N \quad \textbf{(16)}$$

**where** $\boldsymbol{k^{-1}} = \sum_{\boldsymbol{j=0}}^{\infty} \ (\propto -\mathbf{1})\ \boldsymbol{C_j}\ \ (-\mathbf{1})^{\boldsymbol{j}}\ \frac{\boldsymbol{p^j}}{\mathbf{1}-\boldsymbol{p^{(1+j)}}}$.

The plot of Discrete Generalized Exponential distribution is given in Figure.4.

Its survival function and failure rate are

$$\mathbf{S(y) = k}\sum_{\mathbf{j=0}}^{\infty}(-\mathbf{1})^{\mathbf{j}}(\propto -\mathbf{1})\mathbf{C_j}\ \frac{\mathbf{p}^{(\mathbf{1+j})([\mathbf{y}]+\mathbf{j})}}{\mathbf{1} - \mathbf{p}^{(\mathbf{1+j})}}$$

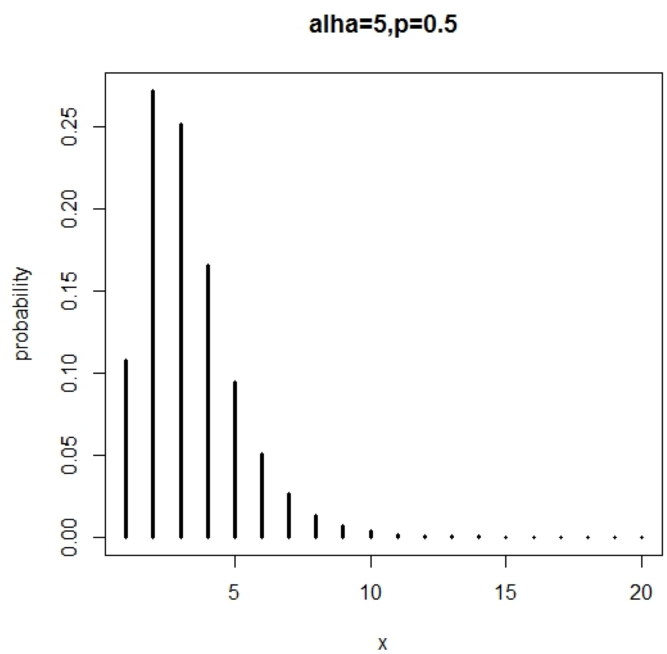


Figure .4. Discrete Generalized Exponential Distribution

And

$$\mathbf{r(y)} = \frac{\mathbf{kp^{y-1}(1 - p^{y})^{\alpha-1}}}{\mathbf{S(y)}}$$

When $\alpha = 1$, failure rate becomes constant. Also when $\alpha = 1$, the generalized exponential reduces to exponential and the corresponding discrete version becomes geometric. Nekoukhou et al. (2012) also used the discrete generalized distribution to model rank frequencies of graphemes in a Slavic language: Slovene.

### 3.2 Discretized distributions using Method 3

In this section we made a review on discrete Rayleigh, Normal, Burr and Pareto distributions.

#### 3.2.1 Discrete Rayleigh Distribution

A Rayleigh random variable X has probability density function given by

$$f(x) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} \;; x \geq 0, \sigma > 0 \quad \textbf{(17)}$$

Also the expressions for distribution function, survival function and hazard function are obtained respectively as

$$\mathbf{F(x) = 1 - e^{\frac{-x^2}{2\sigma^2}}} \;; \mathbf{x \geq 0, \sigma} > 0$$

$$\mathbf{S(x) = e^{\frac{-x^2}{2\sigma^2}}} \;; \mathbf{x \geq 0, \sigma} > 0$$

$$\mathbf{r(x) = \frac{x}{\sigma^2}} \quad ; \quad \mathbf{x \geq 0, \sigma} > 0$$

From which it is clear that Rayleigh distribution has the linearly increasing hazard or failure rate which makes the distribution a possible model for the lifetimes of components that age rapidly with time.
Using method-3, the discrete Rayleigh was obtained by Roy (2004) as

$$\mathbf{P(x) = \theta^{x^2} - \theta^{(x+1)^2}}; \quad \mathbf{x = 0, 1, 2} \ldots \quad (18)$$

The plot of Discrete Rayleigh distribution is given in Figure.5.

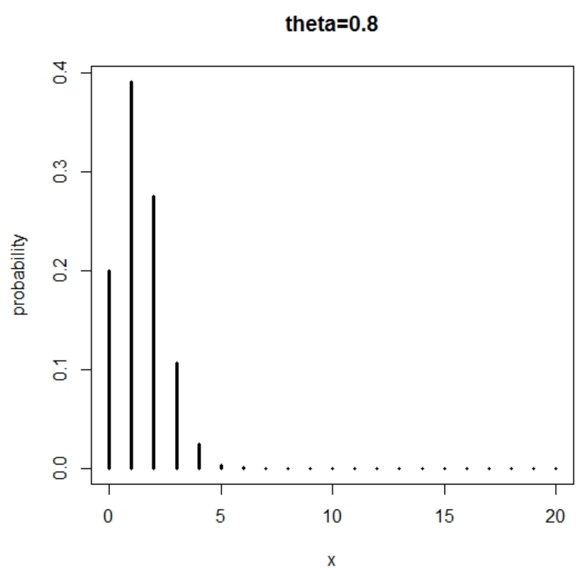


Figure.5 Discrete Rayleigh Distribution

The survival function is given by

$$\mathbf{S(x) = \theta^{x^2} \text{ where } \quad \theta = e^{\frac{-1}{2\sigma^2}}}$$

It describes the survival function of the Rayleigh distribution in the continuous set up so that many reliability properties remain unchanged. Roy (2004) applied this distribution in the reliability determination of a solid shaft, a well-known engineering item.

### 3.2.2 Discrete Normal Distribution

The normal distribution has a remarkable position in probability theory, and can be used as an approximation to many distributions. Applying method-3, Roy (2003) derived discrete normal distribution and its pmf is obtained as

$$\boldsymbol{P(dx = x) = \Phi[\frac{(x+1-\mu)}{\sigma}] - \Phi[\frac{(x-\mu)}{\sigma}] \;= \ldots, -1, 0, 1, \ldots} \quad \textbf{(19)}$$

Where Φ represents the distribution function of the normal deviate Z. For an integer x, the survival function of dx is worked out as

$$S(x) = 1 - \Phi[\frac{(x-\mu)}{\sigma}]$$

Which is same as that of the normal variate X for all integer valued x. As an application of this distribution Roy (2003) considered the problem of reliability determination of a hollow cylinder and elaborated it as an alternative to simulation methods.

The plot of Discrete Normal distribution is given in Figure.6.

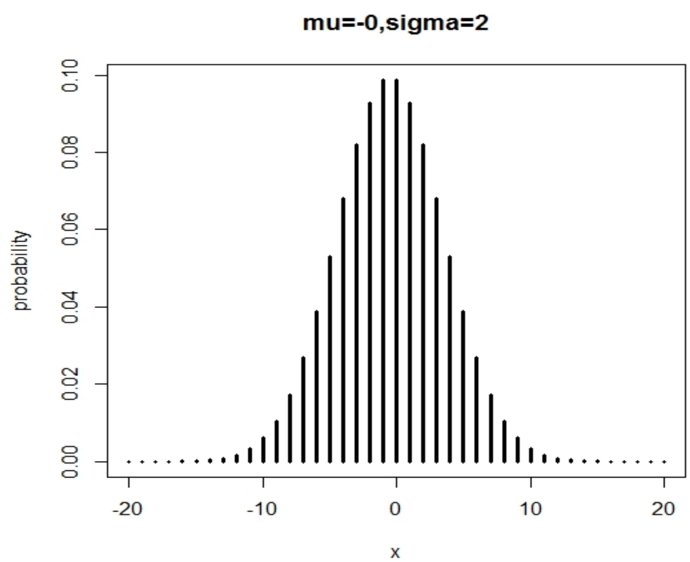


Figure.6. Discrete Normal Distribution

**3.2.3 Discrete Burr and Discrete Pareto Distributions**

Consider the Burr distribution as a continuous lifetime model having probability density function

$$f(x) = \frac{\alpha\beta x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}}; \ \ \alpha > 0, \beta > 0, X > 0 \qquad \textbf{(20)}$$

Its survival function is given by

$$S(x) = (1 + x^{\alpha})^{-\beta} = \theta^{\log(1+x^{\alpha})}$$

Where

$$\theta = e^{(-\beta)} \text{and } 0 < \theta < 1$$

The failure rate becomes

$$r(x) = \frac{f(x)}{S(x)} = \frac{\alpha\beta x^{\alpha-1}}{1 + x^{\alpha}}$$

The second rate of failure is given by

$$r^*(x) = \log\frac{S(x)}{S(x+1)} = -\beta\log\frac{1 + x^{\alpha}}{1 + (1 + x^{\alpha})}$$

Using method-3 Krishna and Pundir (2009) studied discrete Burr model having probability mass function given by

$$P(x) = \theta^{\log[1+(1+x^{\alpha})]} - \theta^{\log[1+(1+x^{\alpha})]} \quad x = 0, 1, 2 \ldots; \tag{21}$$

The plot of Discrete Burr distribution is given in Figure.7.

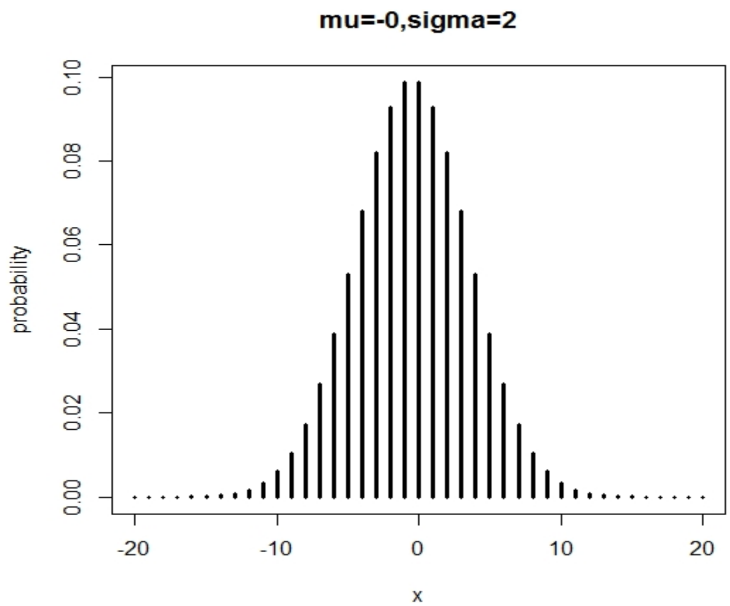


Figure.7. Discrete Burr Distribution

Survival function is

$$S(x) = (1 + x^{\alpha})^{-\beta} = \theta^{\log(1+x^{\alpha})}$$

Where $\theta = e^{(-\beta)}$ and $0 < \theta < 1$

Failure rate is obtained as

$$r(x) = \frac{P(x)}{S(x)} = 1 - \theta^{\phi(x)}$$

Where $\phi(x) = \log[\frac{1+(1+x^{\alpha})}{1+x^{\alpha}}]$

And second rate of failure

$$r^{*}(x) = log\,\frac{S(x)}{S(x+1)} = (log\theta)\log[\frac{1 + x^{\alpha}}{1 + (1 + x^{\alpha})}]$$

S(x) is same for $B(\alpha, \beta)$ and discrete $B(\alpha, \theta)$ at the integer points of X. The expressions for $S(x)$, $r(x)$ and $r^{*}(x)$ for *DBD* $(\alpha, \theta)$ can be directly obtained from those of continuous *Burr* $(\alpha,\beta)$ distribution by putting $\beta=(-log\theta)$.

Pareto distribution is formulated to deal with the distribution of income over a population. From the Burr and discrete Burr distribution for $\alpha = 1$, we get Pareto and discrete Pareto distribution.

The plot of Discrete Pareto distribution is given in Figure.8.

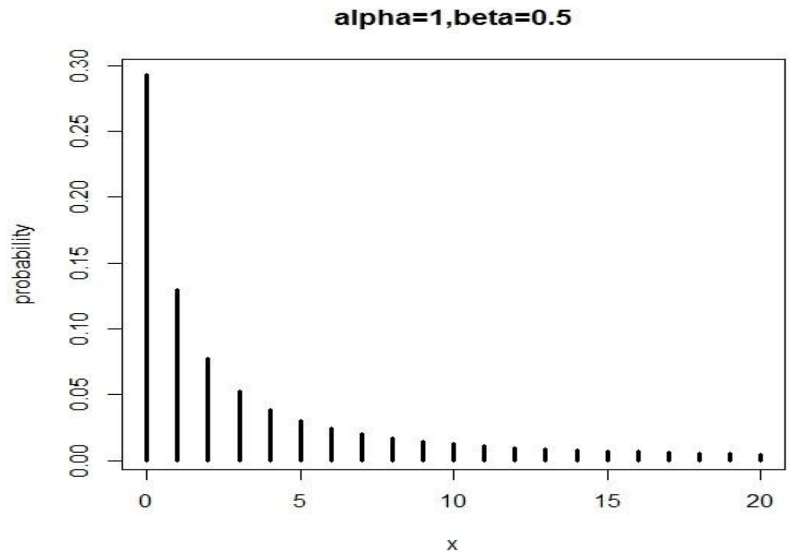

Figure.8. Discrete Pareto Distribution

## 3.3Discretized distributions using Method 5

In this section, we discuss about nabla discrete Gamma distribution and delta discrete Gamma distribution which are derived using method 5.

### 3.3.1 The Nabla Discrete Gamma Distribution

The delta and nabla discrete Gamma distributions are introduced by substituting continuous Taylor monomials and exponential functions with their corresponding discrete types (on the discrete time scale)in continuous gamma distribution. The random variable X has a nabla discrete gamma distribution with parameters ($\alpha,\beta$) if its probability mass function is given by

$$\boldsymbol{P(X=x)=\frac{h^*_{\alpha-1}(x)\beta^{\alpha}}{e^*\beta(\rho(x),0)}=\frac{x^{\alpha-1}\beta^{\alpha}(1-\beta)^{\rho(x)}}{\Gamma(\alpha)}};\ \boldsymbol{\alpha}>0, 0<\beta<1;\quad x=\boldsymbol{N_1} \tag{22}$$

And denote it as $\Gamma^{\nabla}(\alpha,\beta)$ where

$$\mathbf{e^*_{\beta}(\rho(x),0)=(1-\beta)^{-\rho(x)}}$$

Is the nabla exponential function and

$$\boldsymbol{h^*_{\alpha-1}(x)=\frac{x^{\alpha-1}}{\Gamma\alpha}}$$

Is the nabla Taylor monomial.

The mean, variance and moment generating function of the distribution are given by

$$\mathbf{E(X)=\alpha(1-\beta)\beta^{-1}+1}$$
$$\mathbf{V(X)=\alpha(1-\beta)\beta^{-2}}$$
$$\mathbf{M_x(t)=(\frac{1}{1-t(1-\beta)\beta^{-1}})^{\alpha}}$$

Particular cases (a) For $\alpha$=1, $\Gamma^{\nabla}(\alpha,\beta)$ reduces to one parameter nabla discrete gamma or nabla exponential distribution with pmf

$$P(X = x) = \beta(1-\beta)^{p(x)} =; \quad x = 1, 2 \ldots. \tag{23}$$

Obviously, this is the pmf of geometric distribution. (b)For$\alpha = n, n\epsilon N$ , $\Gamma^{\nabla}$ ($\alpha$,$\beta$) reduces to nabla discrete Erlang distribution with pmf

$$P(X = x) = (x - n - 2)C_{x-1}\beta^{n}(1+\beta)^{\rho(x)}; \quad x = N_1 \tag{24}$$

If we substitute $\rho(x) = x$, equations (23) and (24) are given by

$$\mathbf{P(X = x) = \beta(1-\beta)^{x}; x = 0, 1, \ldots.} \tag{25}$$

And

$$P(X = x) = (x + n - 1)C_{x}\beta^{n}(1-\beta)^{x}; \quad x = 0, 1, \ldots \tag{26}$$

Respectively. It can be seen that these equations are the same geometric and same negative binomial distribution. Therefore, we call (25) as nabla geometric distribution and (26) as the nabla negative binomial distribution.

The plot of Nabla discrete Gamma distribution is given in Figure.9.

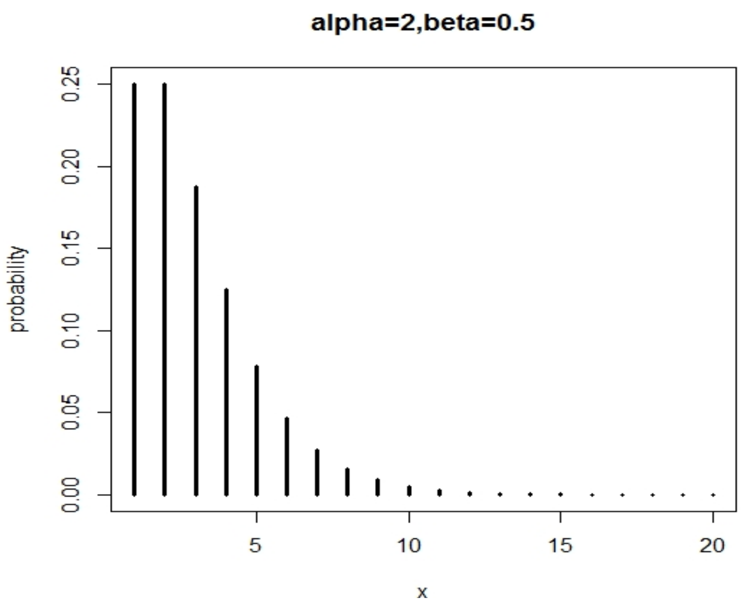


Figure.9. Nabla discrete Gamma distribution

### 3.3.2 The delta discrete Gamma distribution

The random variable X hasa delta discrete gamma distribution with parameters ($\alpha$,$\beta$) if its probability mass function is given by

$$P(X = x) = \frac{h_{\alpha-1}(x)\beta^{\alpha}}{e_{\beta}(\sigma(x),0)} = \frac{x^{\alpha-1}\beta^{\alpha}}{\Gamma(\alpha)(1+\beta)^{\sigma(x)}} \tag{27}$$

Where $\alpha>0$, $\beta>0$, $x=N_{\alpha-1}$ and denote it as $\Gamma^{\Delta}$ ($\alpha$,$\beta$)

$$h_{\alpha-1}(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$$

Is the delta Taylor monomial and

$$\mathbf{e_{\beta}(\sigma(x), 0 = (1+\beta)^{\sigma(x)}}$$

Is the delta exponential function.

Mean, variance and moment generating function of the distribution are given by

$$\mathbf{E(X) = \alpha(1+\beta)\beta^{-1} - 1}$$
$$\mathbf{V(X) = \alpha(1+\beta)\beta^{-2}}$$
$$\mathbf{M_x(t) = (\frac{1}{1 - t(1+\beta)\beta^{-1}})^{\alpha}}$$

The plot of Delta Discrete Gamma distribution is given in Figure.10.

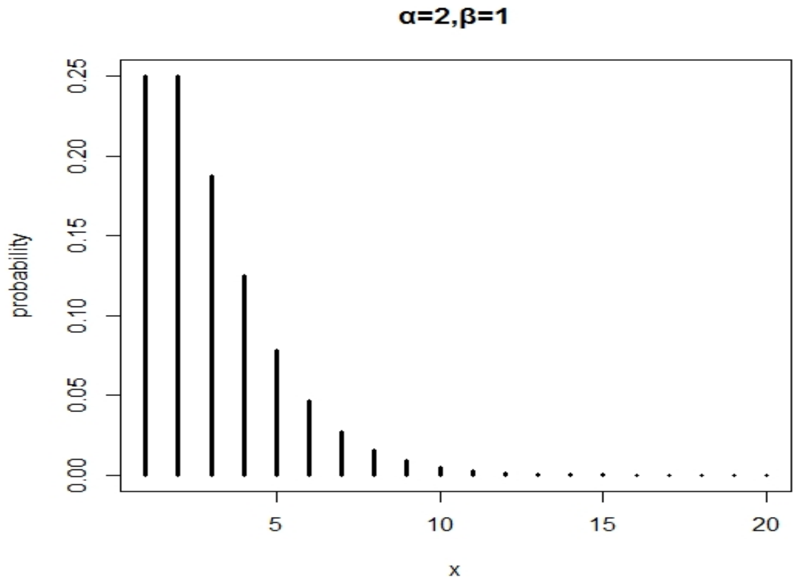


Figure.10. Delta Discrete Gamma Distribution

Particular cases (a) For $\alpha$=1, $\Gamma^{\Delta}(\alpha,\beta)$ reduces to one parameter delta discrete gamma or delta exponential distribution with pmf

$$\boldsymbol{P(X=x) = \beta(1+\beta)^{-}\sigma(x) = (\frac{\beta}{1+\beta})(\frac{1}{1+\beta})^{x}; \qquad x = 0, 1, 2 \ldots.} \tag{28}$$

Obviously, this is the pmf of geometric distribution. (b) For $\alpha=n$, $n\in N$, $\Gamma^{\Delta}(\alpha,\beta)$ reduces to delta discrete Erlang distribution with pmf

$$\boldsymbol{P(X=x) = (x-n+1) = C_x\beta^n(1+\beta)^{-}\sigma(x); \qquad x = N_{n-1}} \tag{29}$$

If we substitute $\sigma(x) = x$, equations (28) and (29) are given by

$$\boldsymbol{P(X=x) = (\frac{\beta}{1+\beta)})(\frac{1}{1+\beta})^{x-1}; \qquad x = 1, 2 \ldots.} \tag{30}$$

And

$$\boldsymbol{P(X=x) = (x-n)C_{x-1}(\frac{1}{1+\beta})^{n}(\frac{1}{1+\beta})^{x-1}; \qquad x = n, n+1, \ldots} \tag{31}$$

Respectively. It can be seen that these equations are the same geometric and same negative binomial distribution. Therefore, we call (30) as delta geometric distribution and (31) as the delta negative binomial distribution.

### 3.4 Discretized distributions using Method 6

In this section the discrete Exponential and Burr distribution of Type I, II and III are reviewed. Moreover discrete gamma and discrete log Cauchy distributions are discussed.

#### 3.4.1 The Discrete Exponential Distribution (Type I)

If X follows exponential distribution with parameter $\theta$, then the pmf of discrete exponential distribution (Type I) obtained by methodology I is

$$P(Y=y)=\begin{cases} w; \quad y=0 \\ we^{-1+e^{-\theta y}}\prod_{i=1}^{y-1}\left(1-e^{-1+e^{\theta i}}\right); \text{ y=1,2, ...m} \\ 0; \quad otherwise; \end{cases} \tag{32}$$

The plot of Discrete Exponential Type I distribution is given in Figure.11.

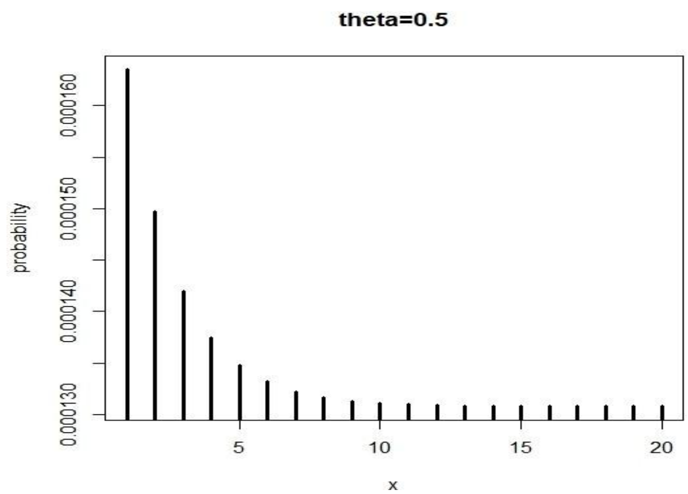


Figure.11.Discrete Exponential Type I Distribution

The hazard rate function of Y is decreasing as that of $X_1$ is decreasing.

#### 3.4.2 Discrete Burr XII Distribution (Type I)

The continuous Burr XII has distribution function given by

$$\mathbf{F_X(x) = 1-(1+x^c)^{-p}}; \quad \mathbf{x} > 0$$

Its pmf obtained by applying methodology I is

$$\boldsymbol{P_Y(y) = we^{(1+y^c)^{-p}-1}\prod_{i=1}^{y-1}\left(1-e^{(1+y^c)^{-p}-1}\right); \ y = 1,2,3 \ldots .m} \tag{33}$$

And hazard rate function is

$$\mathbf{h_Y(y) = e^{(1+y^c)^{-p}-1}; \quad y = 1,2,3 \ldots .m}$$

The plot of Discrete Exponential Type I distribution is given in Figure.12.

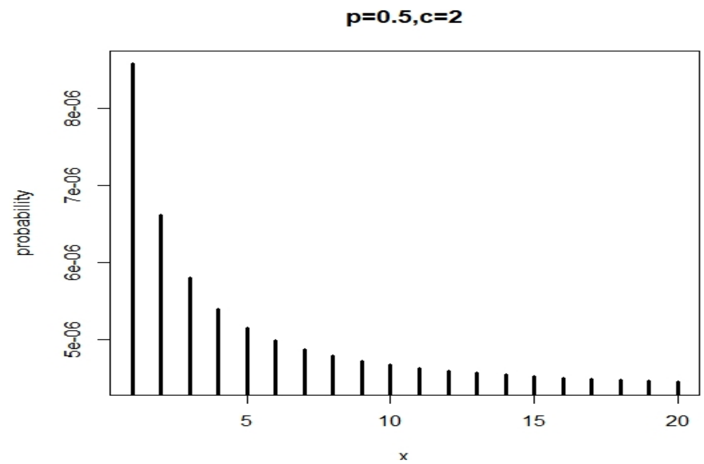


Figure.12. Discrete Burr Type I Distribution

### 3.4.3 Discrete Exponential Distribution (Type II)

If X follows exponential distribution with parameter $\theta$ then the pmf of discrete exponential distribution (Type II) obtained by methodology II is

$$P(Y=y)=\begin{cases} w; \quad y=0 \\ w\frac{2(1-e^{-\theta y})}{2-e^{-\theta y}}\prod_{i=1}^{y-1}\frac{e^{-\theta i}}{2-e^{-\theta i}}; \quad y=1,2,\ldots m \\ 0; \quad otherwise; \end{cases} \tag{34}$$

And hazard rate function is

$$\boldsymbol{h_Y(y)=\frac{2(1-e^{-\lambda y})}{2-e^{-\lambda y}}; \quad y=1,2,3\ldots.m} \tag{35}$$

The plot of Discrete Exponential Type II distribution is given in Figure.13.

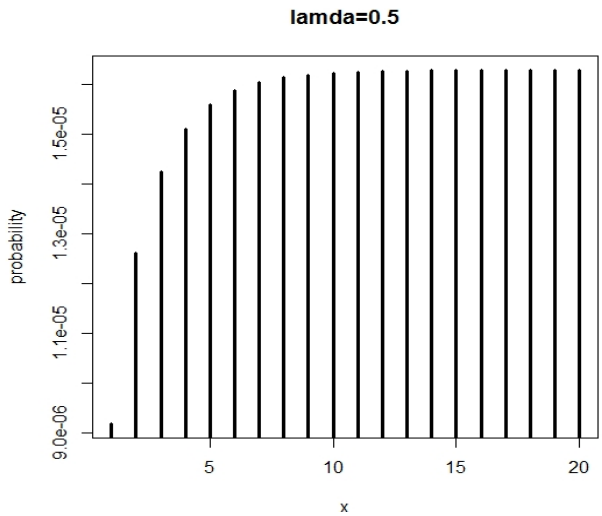


Figure.13. Discrete Exponential Type II Distribution

**3.4.4 Discrete Burr XII Distribution (Type II)**

If X follows Burr XII distribution, then discrete Burr XII obtained by methodology II has the hazard rate function and pmf, respectively, as

$$\mathbf{h_Y(y) = \frac{2(1-(1+y^c)^{-p}}{2-(1+y^c)^{-p}}; \quad y = 1, 2, 3 \ldots . m}$$

$$\boldsymbol{P_Y(y) = [w\frac{2(1-(1+y^c)^{-p}}{2-(1+y^c)^{-p}}]\prod_{i=1}^{y-1}[w\frac{(1+i^c)^{-p}}{2-(1+i^c)^{-p}}]; \quad y = 1, 2, 3 \ldots . m} \qquad \mathbf{(36)}$$

The plot of Discrete Burr Type II distribution is given in Figure.14.

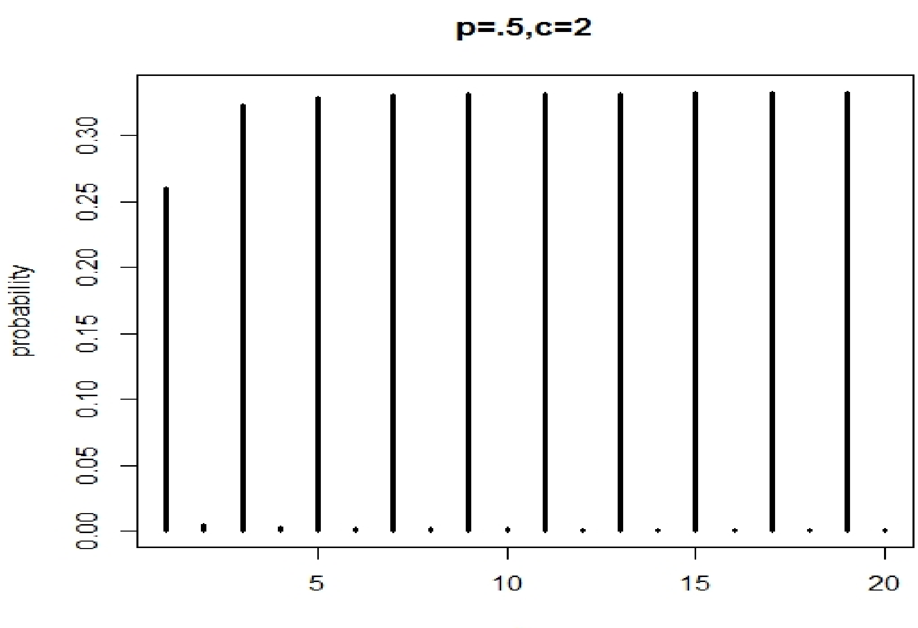


Figure.14.Discrete Burr Type II Distribution

**3.4.5 The Discrete Exponential Distribution (Type III)**

If X follows exponential distribution with parameter $\theta$ then the pmf of discrete exponential distribution (Type III) is obtained by methodology III is

$$P_Y(0) = \frac{w}{\theta + 1}$$

$$P_Y(y) = \frac{w}{\theta_e - \theta_y + 1} \prod_{i=1}^{y-1} \frac{\theta_e - \theta_i}{\theta_e - \theta_i + 1}; \quad y = 1,2,3 \dots .m \tag{37}$$

And its hazard rate function is increasing. The plot of Discrete Exponential Type III distribution is given in Figure.15.

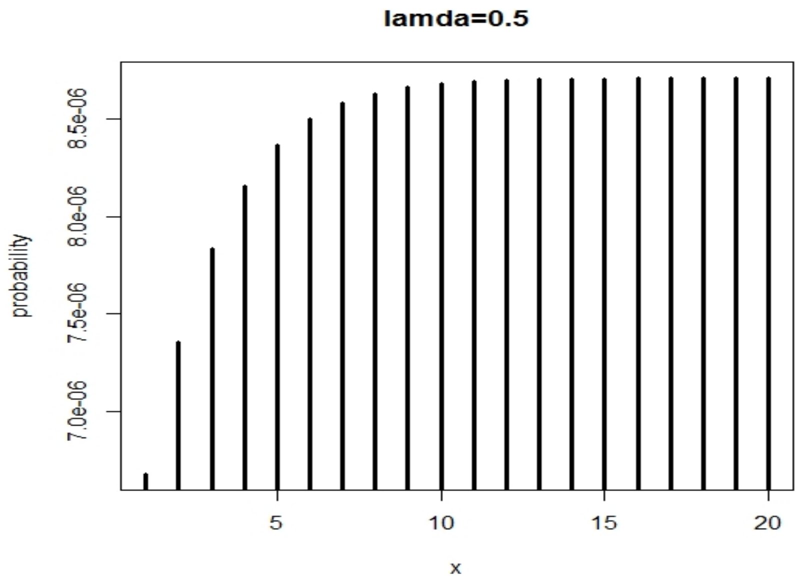


Figure.15. Discrete Exponential Type III Distribution

**3.4.6 The Discrete Gamma Distribution**

If X follows gamma distribution, then the pmf of its discrete version obtained by methodology III is

$$P_Y(y) = w \frac{\Gamma(\alpha)}{\beta^{\alpha} y^{\alpha-1} e^{-\beta y} + \Gamma(\alpha)} \prod_{i=1}^{y-1} \left(1 - \frac{\Gamma(\alpha)}{\beta^{\alpha} j^{\alpha-1} e^{-\beta i} + \Gamma(\alpha)}\right); y = 1,2,3 \dots .m \tag{38}$$

The plot of Discrete Gamma distribution is given in Figure.16

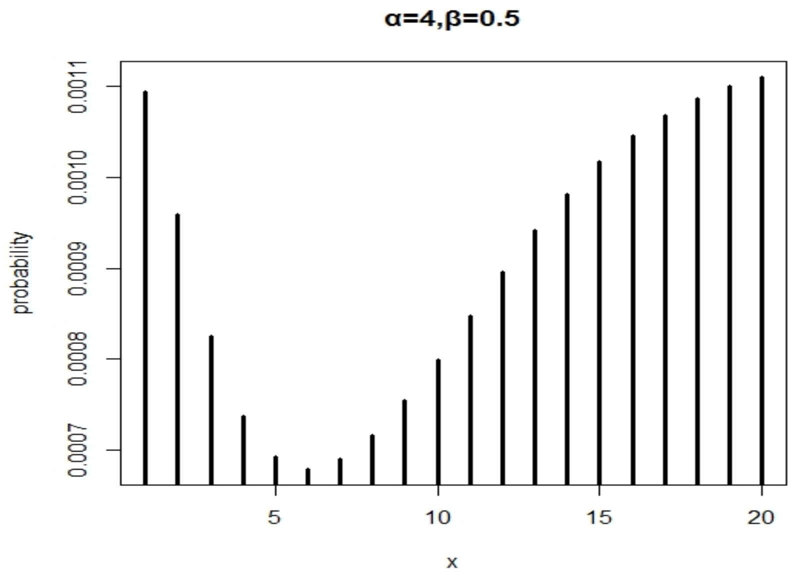


Figure.16. Discrete Gamma Type III Distribution

Its hazard rate function is U- shaped if $\alpha$ greater than 1.

### 3.4.7 Discrete Log Cauchy Distribution

Discrete log Cauchy distribution obtained using methodology III has pmf as

$$P_Y(y) = w\frac{\pi y(lny-\mu)^2+\sigma^2}{\sigma+\pi y(lny-\mu)^2+\sigma^2}\prod_{i=1}^{y-1}\left(\frac{\sigma}{\sigma+\pi i(lni-\mu)^2+\sigma^2}\right); \mathrm{y} = 1,2,3 \ldots.\mathrm{m} \tag{39}$$

And modified unimodal hazard rate function

## 4. Comparison

In this section we made a comparative study on various discretization methods presented in section 2 are done. In method 2, a continuous random variable on R is discretized and the resulting distribution has support on the set of integers. This discrete distribution may not always have a compact form be-cause of the presence of the normalizing constant. In contrast with method 2, the discrete analogue obtained by method 3 has a concise form provided the survival function of the baseline distribution has a concise form. Also this method preserves the survival function. It is to be pointed out in method 4 that if a continuous random variable belongs to the extended exponential family, its discrete version belongs to telescopic family consists of discrete lifetime distributions. Method 5 provides a discretization technique where the pdf to be discretized consists of the monomial Taylor and exponential function. Instead of preserving survival function as in method 2, method 6 preserves the hazard rate function. An important advantage of the method is that discrete analogue so obtained has monotonic and non-monotonic hazard rate functions.

## 5. Summary

We usually come across situations where lifetimes are suitable to measure as discrete random variable rather than continuous one. Geometric and Negative binomial distributions are used to model discrete lifetime data. But the need for more distributions and the availability of various discretization methods lead to a number of discrete distributions suitable to various situations. A review of various discretization methods is carried out in this study. Various discrete models derived by applying these methods are also reviewed. A comparative study of various discretization methods is also done.

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