Maximal and Minimal Positive Continuous Solutions of a Nonlinear Quadratic Integral Equation

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Abstract: In this study, a concernment with the existence of at least one positive continuous solution of the quadratic integral equation

$$x(t) = a(t) + \int_0^t f_1\big(t, s, x(s)\big)ds \int_0^t f_2\big(t, s, x(s)\big), t \in [0.T].$$

The existence of maximal and minimal solutions is also established.

1. Introduction and Preliminaries

Quadratic integral equations (QIEs) arise in several fields, including radiative transfer, kinetic theory of gases, neutron transport, and traffic theory. They have therefore received considerable attention in the literature (see [1]–[14]).

Let $I = [0, T], C = C[0, T]$be the space of continuous functions on $I, and\ L^1 = L^1[0, T]$ be the space of Lebesgue integrable functions on I.

The quadratic integral equation.

$$x(t) = a(t) + \int_0^t f\big(s, x(s)\big)ds \int_0^t g\big(s, x(s)\big)ds \qquad (1)$$

This equation was studied in [10], where the authors proved the existence of at least one continuous solution as well as maximal and minimal solutions.

The quadratic integral equation

$$x(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s)\big)ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\big(s, x(s)\big)ds \ , \alpha, \beta \in (0,1) \qquad (2)$$

This equation was studied in [12], where the authors proved the existence of at least one continuous solution as well as maximal and minimal solutions.

In this study a concernment with the existence of at least one positive continuous solution of the quadratic integral equation

$$x(t) = a(t) + \int_0^t f_1\big(t, s, x(s)\big)ds \int_0^t f_2\big(t, s, x(s)\big)ds \ , \qquad t \in [0\ , T] \qquad (3)$$

The existence of maximal and minimal solutions of the quadratic integral equation (3) will also be established.

The main results are based on the following theorems.

Theorem 1-1: Schauder fixed-point Theorem [15]
Let S be a convex subset of a Banach space B, let the mapping $T: S \to S$ be compact, continuous. Then T has at least one fixed-point in S.

Theorem 1-2: Arzela-Ascoli Theorem [16]
Arzela-Ascoli Theorem let E be a compact metric space and C(E) the Banach space of real or complex valued continuous functions normed by

$$\|f\| = max_{t \in E}|f(t)|$$

If A = {f_n} is a sequence in C(E) such that {f_n} is uniformly bounded and equicontinuous, then the closure of A is compact.

2. Existence of Solutions

Consider the quadratic integral equation (3) under the following assumptions

(i) $a: I = [0, T] \to R_+$ is continuous , $a = sup_{t \in |0,T|}|a(t)|$
(ii) $f_i: I = |0, T| \times |0, T| \times R_+ \to R_+\ are L^1 -$ Caratheodary functions.

i.ef_i measurable in (t , s) for all $x \in R_+$ and continuous in x for almost all $(t,s) \in [0,T] \times [0,T]$ and there exists the functions $m_i(t,s)$ such that

$$|f_i(t,s,x)| \leq m_i(t,s), i = 1,2$$

and $\int_0^t m_i(t,s)ds \leq M_i$, for all $t \in |0,T|$

Moreover f_i are monotonic nonincreasing in $t \in |0,T|$

The following theorem establishes the existence of at least one positive continuous solution of the quadratic integral equation (3).

Theorem 2-1:

If the assumptions (i) and (ii) are satisfied, then the quadratic integral equation(3) has at least one positive solution $x \in C|0,T|$

Proof. Let $C = C|0,T|$ and define the set S by

$$S = \{x \in C: 0 < x \leq r\} \subset C|0,T|$$

Where $r = a + M_1 M_2$

It is clear that S is nonempty, bounded, convex, and closed.

Define the operator F associated with the quadratic integral equation (3) by

$$Fx(t) = a(t) + \int_0^t f_1\big(t,s,x(s)\big)ds \int_0^t f_2\big(t,s,x(s)\big)ds$$

Let $\in S$, then

$$|Fx(t)| = \left| a(t) + \int_0^t f_1\big(t,s,x(s)\big)ds \int_0^t f_2\big(t,s,x(s)\big)ds \right|$$
$$\leq |a(t)| + \int_0^t |f_1(t,s,x(s))|ds \int_0^t |f_2(t,s,x(s))|ds$$
$$\leq |a(t)| + \int_0^t m_1(t,s)ds \int_0^t m_2(t,s)ds$$
$$\leq a + M_1 M_2 = r$$

This proves that $F: S \to S$ and the class of functions $\{F(x)\}$ is uniformly bounded

Let $t_1, t_2 \in [0,T], t_1 < T_2$ and $|t_2 - t_1| \leq \delta$, then

$$|Fx(t_2) - Fx(t_1)| = |a(t_2) - a(t_1)$$
$$+ \int_0^{t_2} f_1\big(t_2,s,x(s)\big)ds \int_0^{t_2} f_2\big(t_2,s,x(s)\big)ds$$
$$- \int_0^{t_1} f_1\big(t_1,s,x(s)\big)ds \int_0^{t_1} f_2(t_1,s,x(s))ds|$$
$$= |a(t_2) - a(t_1)$$
$$+ \left[\int_0^{t_1} f_1\big(t_2,s,x(s)\big)ds + \int_{t_1}^{t_2} f_1\big(t_2,s,x(s)\big)ds\right] \int_0^{t_2} f_2\big(t_2,s,x(s)\big)ds$$
$$- \int_0^{t_1} f_1\big(t_1,s,x(s)\big)ds \int_0^{t_1} f_2(t_1,s,x(s))ds|$$
$$\leq |a(t_2) - a(t_1)|$$
$$+ \left| \int_0^{t_1} f_1\big(t_1,s,x(s)\big)ds \int_0^{t_2} f_2\big(t_1,s,x(s)\big)ds \right.$$
$$+ \int_{t_1}^{t_2} f_1\big(t_1,s,x(s)\big)ds \int_0^{t_2} f_2\big(t_1,s,x(s)\big)ds$$
$$- \int_0^{t_1} f_1\big(t_1,s,x(s)\big)ds \int_0^{t_1} f_2(t_1,s,x(s))ds|$$
$$\leq |a(t_2) - a(t_1)|$$
$$+ \left| \int_0^{t_1} f_1\big(t_1,s,x(s)\big)ds \left[\int_0^{t_2} f_2\big(t_1,s,x(s)\big)ds - \int_0^{t_1} f_2\big(t_1,s,x(s)\big)ds\right] \right.$$

$$+\int_0^{t_2} f_2\big(t_1,s,x(s)\big)ds\int_{t_1}^{t_2} f_1(t_1,s,x(s))ds|$$
$$\le |a(t_2)-a(t_1)|$$
$$+\int_0^{t_1} |f_1(t_1,s,x(s))|ds\int_{t_1}^{t_2} |f_2(t_1,s,x(s))|ds$$
$$+\int_0^{t_2} |f_2(t_1,s,x(s))|ds\int_{t_1}^{t_2} |f_1(t_1,s,x(s))|ds$$
$$\le |a(t_2)-a(t_1)|$$
$$+\int_0^{t_1} m_1(t,s)ds\int_{t_1}^{t_2} m_2(t,s)ds$$
$$+\int_0^{t_2} m_2(t,s)ds\int_{t_1}^{t_2} m_1(t,s)ds$$

This shows that the class of functions is equicontinuous on [0,T]. By the Arzelà–Ascoli theorem [16], F is compact.

Now a prove was made that $F: S\to S$ is continuous let $\{x_n\}\subset S$, and $x_n\to x$, then

$$Fx_n(t)=a(t)+\int_0^t f_1\big(t,s,x_n(s)\big)ds\int_0^t f_2\big(t,s,x_n(s)\big)ds$$
$$\lim_{n\to\infty} Fx_n(t)=\lim_{n\to\infty} a(t)+\lim_{n\to\infty}\left\{\int_0^t f_1\big(t,s,x_n(s)\big)ds\int_0^t f_2\big(t,s,x_n(s)\big)ds\right\}$$

Now

$$f_i(t,s,x_{nk})\to f_i(t,s,x), i=1,2$$

Also

$$|f_i(t,s,x_{nk})|\le m_i(t,s), i=1,2$$

Then, by the Lebesgue dominated convergence theorem [16], we have

$$Fx(t)=\lim_{n_k\to\infty} Fx_{nk}(t)=a(t)+\int_0^t \lim_{n_k\to\infty} f_1\big(t,s,x_{nk}(s)\big)ds\int_0^t \lim_{n_k\to\infty} f_2\big(t,s,x_{nk}(s)\big)ds$$
$$fx(t)=a(t)+\int_0^t f_1\big(t,s,x(s)\big)ds\int_0^t f_2\big(t,s,x(s)\big)ds$$

Then $Fx_n(t)\to Fx(t)$, Which means that the operator F is continuous.

Since all conditions of Schauder fixed point Theorem [15] are satisfied, then the operator F has at least one fixed point $x\in C|0,T|$, which completes the proof .

3. Existence of the maximal and minimal solutions

Definition 3-1:

Let *q(t)* be a solution of the quadratic integral equation (1) . Then *q(t)* is said to be a maximal solution of (1) if every solution *x(t)* of (1) satisfies the inequality (see [17]) .

$$x(t)<q(t). \quad t\in[0,T] \qquad (4)$$

A minimal solution s(t) is defined similarly by reversing the above inequality, i.e.,

$$x(t)>s(t), \; t\in[0,T] \qquad (5)$$

Consider the following lemma.

Lemma 3-2:

Let $f_i(t,s,x)$ satisfy the assumption (ii) and *x(t)*, *y(t)* are two continuous functions satisfying

$$x(t)\le a(t)+\int_0^t f_1\big(t,s,x(s)\big)ds\int_0^t f_2\big(t,s,x(s)\big)ds\,,\;\; t\in[0,T]$$
$$y(t)\ge a(t)+\int_0^t f_1\big(t,s,y(s)\big)ds\int_0^t f_2\big(t,s,y(s)\big)ds\,,\;\; t\in[0,T]$$

And at least one of the inequalities is strict.

If f_i , $i = 1{,}2$ are monotonic nondecreasing in x, then

$$x(t) < y(t), \ \ t > 0 \qquad (6)$$

Proof. Suppose that conclusion (6) is false. Then there exists t_1 such that

$$r(t_1) = y(t_1)\,, \ \ t_1 > 0$$

And

$$x(t) < y(t), \ \ 0 < t < t_1$$

From the monotonicity of f_1, f_2 in *x*, we get

$$\begin{aligned} x(t_1) &\le a(t_1) + \int_0^{t_1} f_1\big(t_1, s, x(s)\big)ds \int_0^{t_1} f_2\big(t_1, s, x(s)\big)ds \ , \ \ t \in [0,T] \\ &< a(t_1) + \int_0^{t_1} f_1\big(t_1, s, y(s)\big)ds \int_0^{t_1} f_2\big(t_1, s, y(s)\big)ds \ , \qquad t \in [0,T] \\ & x(t_1) < y(t_1) \end{aligned}$$

Which contradicts the fact that $x(t_1) = y(t_1)$.

Then

$$x(t) < y(t)$$

The following theorem establishes the existence of continuous maximal and minimal solutions of the quadratic integral equation (3).

Theorem 3-3:

Let the assumptions *(i)* and *(ii)* of Theorem 2-1 are satisfied. If $f_1(t,s,x), f_2(t,s,x)$ are monotonic nondecreasing in *x* for each $\in [0,T]$. then the quadratic integral equation (3) has maximal and minimal solutions .

Proof. First, we prove the existence of the maximal solution of (3).

Let $\in > 0$ be given, and consider the quadratic integral equation

$$x_\in(t) \le a(t) + \int_0^t f_{1\in}\big(t, s, x_\in(s)\big)ds \int_0^t f_{2\in}\big(t, s, x_\in(s)\big)ds \ , \qquad t \in [0,T] \qquad (7)$$

Where

$$f_{i\in}\big(t, s, x_\in(s)\big) = f_i\big(t, s, x_\in(s)\big) + \in , \qquad i = 1{,}2$$

Clearly the function $f_{i\in}\big(t,s,x_\in(s)\big), i = 1{,}2 \ are \ L^{1-}$ Caratheodary functions, therefore the equation (7) has a solution on *C[0,T]*

Let $\in_1$, $\in_2$ be such that $0 < \in_2 < \in_1 < \in$, then

$$\begin{aligned} x_{\in 2}(t) &= a(t) + \int_0^t f_{1\in 2}\big(t, s, x_{\in 2}(s)\big)ds \int_0^t f_{2\in 2}\big(t, s, x_{\in 2}(s)\big)ds \\ &= a(t) + \int_0^t \big(f_1\big(t, s, x_{\in 2}(s)\big) + \in 2\big)ds \int_0^t \big(f_2\big(t, s, x_{\in 2}(s)\big) + \in 2\big)ds \qquad (8) \end{aligned}$$

Also

$$\begin{aligned} x_{\in 1}(t) &= a(t) + \int_0^t f_{1\in 1}\big(t, s, x_{\in 1}(s)\big)ds \int_0^t f_{2\in 1}\big(t, s, x_{\in 1}(s)\big)ds \\ &= a(t) + \int_0^t (f_1\big(t, s, x_{\in 1}(s)\big) + \in 1 ds \int_0^t \big(f_2\big(t, s, x_{\in 1}(s)\big) + \in 1\big)ds \\ x_{\in 1}(t) &> a(t) + \int_0^t f_1\big(t, s, x_{\in 1}(s)\big)ds \int_0^t f_2\big(t, s, x_{\in 1}(s)\big)ds \ \ (9) \end{aligned}$$

Applying Lemma 3-2 on (8) and (9) we have

$$x_{\in 2}(t) < x_{\in 1}(t) \qquad for \ t \in [0,T]$$

As shown before, the family of functions $x_\in(t)$ is equi-continuous and uniformly bounded.

Hence, by Arzela – Ascoli Theorem (see [16]), there exists a decreasing sequence $\in_n$ such that $\in_n \to 0 as n \to \infty$, and $\lim_{n\to\infty} x_{\in n}(t)$ exists uniformly in [0 , T] and denote the limit by *q(t)* . From the continuity of the functions $f_{i\in}\big(t,s,x_\in(s)\big)$, $i = 1,2$ in the third argument, here get

$$f_{i\in}\big(t,s,x_\in(s)\big) \to f_i\big(t,s,(s)\big) \qquad as\ n \to \infty\ , i = 1,2$$

And

$$q(t) = \lim_{n\to\infty} x_{\in n}\ (t) = (t) + \int_0^t f_1\big(t,s,q(s)\big)ds \int_0^t f_2\big(t,s,q(s)\big)ds$$

Which implies that q(t) is a solution of the quadratic integral equation (3).

Finally, we show that q(t) is the maximal solution of (3).

Let x(t) be any solution of (3). Then

$$x(t) = a(t) + \int_0^t f_1\big(t,s,x(s)\big)ds \int_0^t f_2\big(t,s,x(s)\big)ds \qquad (10)$$

Also

$$x,(t) = a(t) + \int_0^t f_{i\in}\big(t,s,x_\in(s)\big)ds \int_0^t f_{2\in}\big(t,s,x_\in(s)\big)ds$$

$$x_\in(t) > a(t) + \int_0^t f_1\big(t,s,x_\in(s)\big)ds \int_0^t f_2\big(t,s,x_\in(s)\big)ds \qquad (11)$$

Applying Lemma 3-2

On (10) and (11) we get

$$x(t) < x,(t), \qquad for\ t \in [0,T]$$

From the uniqueness of the maximal solution (see [17]), it is clear that *x,(t)* tends to *q(t)* uniformly in [0 , T] as $\in \to \infty$.

Similarly, the existence of the minimal solution can be proved. We set

$$f_{i\in}\big(t,s,x_\in(s)\big) = f_i\big(t,s,x_\in(s)\big) - \in , i = 1,2$$

And establish the existence of a minimal solution.

4. Applications

As an application, we consider the quadratic integral equation of fractional order (3).
We can now state the following theorems.

Theorem 4-1:

Assume that *f.g*: $[0,T]xR_+ \to R_+ are L^{1-}$ Caratheodary functions with $|f| \le m_1\ and\ |g| \le m_2$, then the quadratic integral equation (3) has at least one positive continuous solution .

Theorem 4-2:

Let the assumptions of theorem 4-1 are satisfied. Assume that the functions f and g are nondecreasing in the second argument, then the quadratic integral equation (3) has maximal and minimal solution.

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