International Journal of Latest Research in Engineering and Management" (IJLREM)
ISSN: 2456-0766
www.ijlrem.org ‖ Volume 04 Issue 09 ‖ September 2020 ‖ PP 01-06

Some delay integral inequalities and applications

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Abstract: In this paper, we establish some new types of delay integral inequalities with two independent variables on time scales. The inequalities can be used in the investigation of qualitative properties of delay dynamic equations. We also propose some applications for them.

Keywords: integral inequality; time scales; dynamic equation; qualitative properties.

I. INTRODUCTION

The development of the theory of time scales was initiated by Hilgewr [1], and the purpose of the theory of time scales is to unify continuous and discrete analysis. A time scale is an arbitrary nonempty closed subset of the real numbers. Many integral inequalities on time scales have been established since then, for example [2-11], which have been designed in order to unify continuous and discrete analysis. But to our knowledge, the delay integral inequalities on time scales have been scarcely payed attention to in the literature so far [12,13], and furthermore, nobody has studied the delay integral inequalities with two independent variables on time scales.

Our aim in this paper is to establish some new delay integral inequalities with two independent variables on time scales, and present some applications for them. For two given sets G, H, we denote the set of maps from G to H by (G,H), while denote the definition domain and the image of a function f by $Dom(f)$ and $\mathrm{Im}(f)$ respectively.

In the rest of the paper, R denotes the set of real numbers and $R_+=[0,\infty)$. T denotes an arbitrary time scale and $T_0=[x_0,\infty)\bigcap T, \overline{T}_0=[y_0,\infty)\bigcap T$, where $x_0, y_0\in T$. The set T^κ is defined to be T if T does not have a left-scattered maximum, otherwise it is T without the left-scattered maximum. On T we define the forward and backward jump operators $\sigma(t)\in(T,T)$ and $\rho(t)\in(T,T)$ such that $\sigma(t)=\inf\{s\in T, s>t\}$, $\rho(t)=\sup\{s\in T, s<t\}$.

Definition 1: A point $t\in T$ with $t>infT$ is said to be left-dense if $\rho(t)=t$ and right-dense if $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$.

Definition 2: A function $f\in(T,R)$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1+\mu(t)f(t)\neq 0$, where $\mu(t)=\sigma(t)-t$. C_{rd} denotes the set of rd-continuous functions, while $\overline{R}$ denotes the set of all regressive and rd-continuous functions, and

$$\overline{R}^+=\{f\mid f\in\overline{R}, 1+\mu(t)f(t)>0, \forall t\in T\}.$$

Definition 3: For some $t\in T^\kappa$, and a function $f\in(T,R)$, the delta derivative of f is denoted by $f^\Delta(t)$, and satisfies

$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t)-s)| \le \varepsilon |\sigma(t)-s|$ for $\forall \varepsilon > 0$, where $s \in U$, and U is a neighborhood of t. The function f is called delta differential on T^{κ}

Similarly, for some $y \in T^{\kappa}$, and a function $f \in (T \times T, R)$,

the partial delta derivative of f with respect to y is denoted by $(f(x,y))_y^{\Delta}$, and satisfies

$|f(x,\sigma(y)) - f(x,s) - (f(x,y))_y^{\Delta}(\sigma(y)-s)| \le \varepsilon |\sigma(y)-s|$ for $\forall \varepsilon > 0$, where $s \in U$, and U is a neighborhood of y.

Definition 4: For some $a, b \in T$ and a function $f \in (T, R)$, the Cauchy integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $F^{\Delta}(t) = f(t), t \in T^{\kappa}$.

Similarly, for some $a, b \in T$ and a function $f \in (T \times T, R)$, the Cauchy partial integral of f with respect to y is defined by

$$\int_a^b f(x,y)\Delta y = F(x,b) - F(x,a),$$

where $(F(x,y))_y^{\Delta} = f(x,y), y \in T^{\kappa}$

More details on time scales can be referred to [14]..

II. Main Results

We will give some lemmas for further use.

Lemma 2.1 ([14], Gronwall0s inequality): Suppose $X \in T_0$ is an arbitrarily fixed number, and $u(X,y)$, $b(X,y) \in C_{rd}$, $m(X,y) \in \overline{R}^{+}$ with respect to y, $m(X,y) \ge 0$, then

$$u(X,y) \le b(X,y) + \int_{y_0}^{y} m(X,t)u(X,t)\Delta t, y \in \overline{T}_0$$

implies

$$u(X,y) \le b(X,y) + \int_{y_0}^{y} e_m(y,\sigma(t))b(X,t)m(X,t)\Delta t, y \in \overline{T}_0$$

where $e_m(y, y_0)$ is the unique solution of the following equation

$$(z(X,y))_y^{\Delta} = m(X,y)z(X,y), z(X,y_0) = 1.$$

Lemma 2.2 [15]: Assume that $a \ge 0, p \ge q \ge 0$, and $p \ne 0$, then for any $K > 0$,

$$a^{\frac{q}{p}} \le \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

Theorem 2.1: Suppose $u, f, g, h \in C_{rd}(T_0 \times \overline{T}_0, R_+)$, p, q, r

m, C are constants, and $p \ge q \ge 0, p \ge r \ge 0, p \ge m \ge 0,$
$p \ne 0, C > 0$, $\tau_1 \in (T_0, T), \tau_1(x) \le x, -\infty < \alpha = \inf\{\tau_1(x), x \in T_0\} \le x_0$,

$\tau_2 \in (\overline{T}_0, T), \tau_2(y) \le y, -\infty < \beta = \inf\{\tau_2(y), y \in \overline{T}_0\} \le y_0$,

$\phi \in C_{rd}(([\alpha, x_0]\times[\beta, y_0])\bigcap T^2, R_+)$, $K > 0$ is an arbitrary constant. If for $(x,y)\in(T_0,\overline{T_0})$, $u(x,y)$ satisfies the following inequality

$$u^p(x,y) \le C + \int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)u^q(\tau_1(s),\tau_2(t)) + g(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t + \int_{y_0}^{y}\int_{x_0}^{x}\int_{y_0}^{t}\int_{x_0}^{s} h(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t \tag{1}$$

with the initial condition

$$\begin{cases} u(x,y) = \phi(x,y), x\in[\alpha,x_0]\bigcap T, or, y\in[\beta,y_0]\bigcap T \\ \phi(\tau_1(x),\tau_2(y)) \le C^{\frac{1}{p}}, \tau_1(x)\le x_0, or, \tau_2(y)\le y_0 \end{cases}, \tag{2}$$

then

$$u(x,y) \le [B_1(x,y) + \int_{y_0}^{y} e_{B_2}(y,\sigma(t))B_2(x,t)B_1(x,t)\Delta t]^{\frac{1}{p}},\ (x,y)\in(T_0,\overline{T_0}) \tag{3}$$

where

$$B_1(x,y) = C + \int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)\frac{p-q}{p}K^{\frac{q}{p}} + g(s,t)\frac{p-r}{p}K^{\frac{r}{p}} + \int_{y_0}^{t}\int_{x_0}^{s} h(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$B_2(x,y) = \int_{x_0}^{x}[f(s,y)\frac{q}{p}K^{\frac{q-p}{p}} + g(s,y)\frac{r}{p}K^{\frac{r-p}{p}} + \int_{y_0}^{y}\int_{x_0}^{s} h(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta]\Delta s \cdot$$

Proof : Given a fixed $X\in T_0$, and $x\in[x_0,X]\bigcap T, y\in\overline{T_0}$. Let the right side of (1) be $v(x,y)$, then

$$u(x,y) \le v^{\frac{1}{p}}(x,y) \le v^{\frac{1}{p}}(X,y), \forall x\in[x_0,X]\bigcap T, y\in\overline{T_0} \tag{4}$$

If $\tau_1(x)\ge x_0$ and $\tau_2(y)\ge y_0$, then

$\tau_1(x)\in[x_0,X]\bigcap T, \tau_2(y)\in\overline{T_0}$, and

$$u(\tau_1(x),\tau_2(y)) \le v^{\frac{1}{p}}(\tau_1(x),\tau_2(y)) \le v^{\frac{1}{p}}(x,y) \tag{5}$$

If $\tau_1(x)\le x_0$ or $\tau_2(y)\le y_0$, then from (2) we have

$$u(\tau_1(x),\tau_2(y)) = \phi(\tau_1(x),\tau_2(y)) \le C^{\frac{1}{p}} \le v^{\frac{1}{p}}(x,y) \tag{6}$$

From (5) and (6) we always have

$$u(\tau_1(x),\tau_2(y)) \le v^{\frac{1}{p}}(x,y), x\in[x_0,X]\bigcap T, y\in\overline{T_0} \tag{7}$$

So

$$v(X,y)=C+\int_{y_0}^{y}\int_{x_0}^{X}[f(s,t)u^q(\tau_1(s),\tau_2(t))+g(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)u^m(\tau_1(\xi),\tau_2(\eta))\Delta\xi\Delta\eta\Delta s\Delta t$$

$$\leq C+\int_{y_0}^{y}\int_{x_0}^{X}[f(s,t)v^{\frac{q}{p}}(s,t)+g(s,t)v^{\frac{r}{p}}(s,t)]\Delta s\Delta t+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)v^{\frac{m}{p}}(\xi,\eta)\Delta\xi\Delta\eta\Delta s\Delta t \tag{8}$$

From Lemma 2.2, for $\forall K>0$, we have

$$\begin{cases} v^{\frac{q}{p}}(x,y)\leq \dfrac{q}{p}K^{\frac{q-p}{p}}v(x,y)+\dfrac{p-q}{p}K^{\frac{q}{p}} \\ v^{\frac{r}{p}}(x,y)\leq \dfrac{r}{p}K^{\frac{r-p}{p}}v(x,y)+\dfrac{p-r}{p}K^{\frac{r}{p}} \\ v^{\frac{m}{p}}(x,y)\leq \dfrac{m}{p}K^{\frac{m-p}{p}}v(x,y)+\dfrac{p-m}{p}K^{\frac{m}{p}} \end{cases} \tag{9}$$

Combining (8), (9) we have

$$v(X,y)\leq C+\int_{y_0}^{y}\int_{x_0}^{X}[f(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)+\frac{p-q}{p}K^{\frac{q}{p}})$$

$$+g(s,t)(\frac{r}{p}K^{\frac{r-p}{p}}v(s,t)+\frac{p-r}{p}K^{\frac{r}{p}})]\Delta s\Delta t$$

$$+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)(\frac{m}{p}K^{\frac{m-p}{p}}v(\xi,\eta)+\frac{p-m}{p}K^{\frac{m}{p}})\Delta\xi\Delta\eta\Delta s\Delta t$$

$$\leq C+\int_{y_0}^{y}\int_{x_0}^{X}[f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}+g(s,t)\frac{p-r}{p}K^{\frac{r}{p}}\Delta s\Delta t\quad+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)\frac{p-m}{p}K^{\frac{m}{p}}\Delta\xi\Delta\eta]\Delta s\Delta t$$

$$+\int_{y_0}^{y}\{\int_{x_0}^{X}[f(s,t)\frac{q}{p}K^{\frac{q-p}{p}}+g(s,t)\frac{r}{p}K^{\frac{r-p}{p}}+\int_{y_0}^{y}\int_{x_0}^{X}\int_{y_0}^{t}\int_{x_0}^{s}h(\xi,\eta)\frac{m}{p}K^{\frac{m-p}{p}}\Delta\xi\Delta\eta\Delta s\}v(X,t)\Delta t$$

$$=B_1(X,y)+\int_{y_0}^{y}B_2(X,t)v(X,t)\Delta t \tag{10}$$

From Lemma 2.1 we have

$$u(x,y)\leq B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)B_1(X,t)\Delta t,\ y\in\overline{T_0} \tag{11}$$

Then combining (4), (11) we obtain

$$u(x,y)\leq[B_1(X,y)+\int_{y_0}^{y}e_{B_2}(y,\sigma(t))B_2(X,t)B_1(X,t)\Delta t]^{\frac{1}{p}},\ x\in[x_0,X]\cap T,y\in\overline{T_0} \tag{12}$$

Take $x=X$, then it follows

$$u(X,y) \le [B_1(X,y) + \int_{y_0}^{y} e_{B_2}(y,\sigma(t)) B_2(X,t) B_1(X,t)\Delta t]^{\frac{1}{p}} \tag{13}$$

Considering $X \in T_0$ is arbitrary, substituting X with x we can obtain the desired inequality (3).

Remark 1: If we take $T = R, p = q = 1, g(x,y) = h(x,y) \equiv 0$, then Theorem 2.1 reduces to [16, Theorem 2.2], which is one case of integral inequality for continuous function.

Remark 2: If we take $T = Z, p = q, a(x,y) \equiv C, b(x,y) \equiv 1, h(x,y) \equiv 0$,
then the Theorem 2.1 reduces to [17, Corollary 2.6], which is one case of discrete inequality..

III. APPLICATIONS

In this sections, we will present some applications for the results we have established above, and try to give explicit bounds for solutions of certain dynamic equations.

Example 1: Consider the following delay dynamic differen-tial equation

$$(u^p(x,y))_{yx}^{\Delta\Delta} = F(x,y,u(\tau_1(x),\tau_2(y))), (x,y) \in (T_0,\overline{T_0}) \tag{14}$$

with the initial condition

$$\begin{cases} u(x,y) = \phi(x,y), x \in [\alpha, x_0] \bigcap T, or, y \in [\beta, y_0] \bigcap T \\ |\phi(\tau_1(x),\tau_2(y))| \le |C|^{\frac{1}{p}}, \tau_1(x) \le x_0, or, \tau_2(y) \le y_0 \end{cases}, \tag{15}$$

where $u \in C_{rd}(T_0 \times \overline{T}_0, R), p > 0$ is a constant, $C = u^p(x_0, y_0)$

$\phi \in C_{rd}(([\alpha, x_0] \times [\beta, y_0]) \bigcap T^2, R)$, $\alpha, \beta, \tau_1, \tau_2$ are the same as in Theorem 2.1.

Theorem 3.1: Suppose $u(x,y)$ is a solution of (14), and $|F(x,y,u)| \le f(x,y)|u|^q + g(x,y)|u|^r$, where f, g, q, r are
defined the same as in Theorem 2.1, then

$$|u(x,y)| \le [B_1(x,y) + \int_{y_0}^{y} e_{B_2}(y,\sigma(t)) B_2(x,t) B_1(x,t)\Delta t]^{\frac{1}{p}},\ (x,y) \in (T_0,\overline{T_0}) \tag{16}$$

where

$$B_1(x,y) = |C| + \int_{y_0}^{y}\int_{x_0}^{x} [f(s,t)\frac{p-q}{p} K^{\frac{q}{p}} + g(s,t)\frac{p-r}{p} K^{\frac{r}{p}}]\Delta s \Delta t$$

$$B_2(x,y) = \int_{x_0}^{x} [f(s,y)\frac{q}{p} K^{\frac{q-p}{p}} + g(s,y)\frac{r}{p} K^{\frac{r-p}{p}}]\Delta s$$

Proof: The equivalent integral equation of (14) can be denoted by

$$u^p(x,y) = C + \int_{y_0}^{y}\int_{x_0}^{x} F(s,t,u(\tau_1(s),\tau_2(t)))\Delta s \Delta t \tag{17}$$

Then

$$|u^p(x,y)| \le |C| + \int_{y_0}^{y}\int_{x_0}^{x} |F(s,t,u(\tau_1(s),\tau_2(t)))| \Delta s \Delta t$$

$$\le |C| + \int_{y_0}^{y}\int_{x_0}^{x} [f(s,t)u^q(\tau_1(s),\tau_2(t)) + g(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s \Delta t,$$

and a suitable application of Theorem 2.1 yields the desired inequality (16).

REFERENCES

[1] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18-56..

[2] F.H. Wong, C.C. Yeh, S.L. Yu, C.H. Hong, Young¡¯s inequality and related results on time scales, Appl. Math. Lett. 18 (2005) 983-988.

[3] F.H. Wong, C.C. Yeh, W.C. Lian, An extension of Jensen¡¯s inequality on time scales, Adv. Dynam. Syst. Appl. 1 (1) (2006) 113-120.

[4] M.Z. Sarikaya, On weighted Iyengar type inequalities on time scales, Applied Mathematics Letters, 22 (2009) 1340-1344.

[5] W.J. Liu, C.C. Li, Y.M. Hao, Further generalization of some double integral inequalities and applications, Acta. Math. Univ. Comenian. 77 (1) (2008) 147-154.

[6] X.L. Cheng, Improvement of some Ostrowski-Gruss type inequalities, Comput. Math. Appl. 42 (2001) 109-114.

[7] M. Bohner, T. Matthews, The Gruss inequality on time scales, Commun. Math. Anal. 3 (1) (2007) 1-8.

[8] Q.A. Ngo, Some mean value theorems for integrals on time scales, Applied Mathematics and Computation, 213 (2009) 322-328.

[9] W.j. Liu, Q.A. Ngo, Some Iyengar-type inequalities on time scales for functions whose second derivatives are bounded, Applied Mathematics and Computation, 216 (2010) 3244-3251.

[10] W.j. Liu, Q.A. Ngo, A generalization of Ostrowski inequality on timescales for k points, Applied Mathematics and Computation, 203 (2008) 754-760.

[11] R. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: a survey, Math. Inequal. Appl. 4 (4) (2001) 535-557.

[12] W.N. Li, Some delay integral inequalities on time scales, Computers and Mathematics with Applications, 59 (2010) 1929-1936.

[13] Qing-Hua Maa, Josip Pecaric, The bounds on the solutions of certain two-dimensional delay dynamic systems on time scales, (Article In Press), Computers and Mathematics with Applications.

[14] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhauser, Boston, 2001.

[15] Fangcui Jiang, Fanwei Meng, Explicit bounds on some new nonlinear integral inequality with delay, J. Comput. Appl. Math. 205 (2007) 479-486.

[16] Hongxia Zhang, Fanwei Meng, Integral inequalities in two independent variables for retarded Volterra equations, Applied Mathematics and Computation 199 (2008) 90-98.

[17] Wing-Sum Cheung, Jingli Ren, Discrete non-linear inequalities and applications to boundary value problems, J. Math. Anal. Appl. 319 (2006) 708-724.