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Research on Exact Solution for Nonlinear Heat Conduction Equation

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Abstract: In this paper, we derive exact traveling wave solutions of nonlinear heat conduction equation by a presented method. The method appears to be efficient in seeking exact solutions of nonlinear equations.

Keywords: (G'/G)-expansion method, traveling wave solutions, exact solution, heat conduction equation.

I. INTRODUCTION

In scientific research, seeking the exact solutions of nonlinear equations is a hot topic. Many approaches have been presented so far [1-6]. In [7], Mingliang Wang proposed a new method called (G'/G)-expansion method. The main merits of the (G'/G)-expansion method over the other methods are that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset. So the application of the (G'/G)-expansion method attracts many author's attention. Our aim in this paper is to present an application of the (G'/G)-exp-ansion method to nonlinear heat conduction equation.

II. DESCRIPTION OF THE (G'/G)-EXPANSION METHOD

In this section we will describe the (G'/G)-expansion method for finding out the traveling wave solutions of no-nlinear evolution equations.

Suppose that a nonlinear equation, say in three indep-endent variables x, y and t , is given by

$$P(u,u_{t,}u_x,u_y,u_{tt},u_{xt},u_{yt},u_{xx},u_{yy}......)=0 \tag{2.1}$$

where u = u(x, y, t) is an unknown function, P is a polyno-mial in u = u(x, y, t) and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the (G'
/G)-expansion method.

Step 1. Combining the independent variables x, y and t into one variable $\xi=\xi(x,y,t)$, we suppose that

$$u(x,y,t)=u(\xi),\xi=\xi(x,y,t) \tag{2.2}$$

the travelling wave variable (2.2) permits us reducing Eq. (2.1) to an ODE for $u=u(\xi)$

$$P(u,u',u'',......)=0 \tag{2.3}$$

Step 2. Suppose that the solution of (2.3) can be expre-ssed by a polynomial in (G'/G) as follows:

$$u(\xi) = \alpha_m (\tfrac{G'}{G})^m + ... \tag{2.4}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{2.5}$$

$\alpha_m, ...\lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$ .The unwritten part in (2.4) is also a polynomial in $(\frac{G'}{G})$, the degree of which is generally equal to or less than m−1. The positive integer m can be determined by consider-ing the homogeneous balance between the highest order de-rivatives and nonlinear terms appearing in (2.3).

Step 3. Substituting (2.4) into (2.3) and using second
order LODE (2.5), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of Eq. (2.3) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equation-ns for $\alpha_m, ...\lambda$ and μ .

Step 4. Assuming that the constants $\alpha_m, ...\lambda$ and μ Can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (2.5)
have been well known for us, substituting $\alpha_m, ...$ and the general solutions of Eq. (2.5) into (2.4) we can obtain the traveling wave solutions of the nonlinear evolution equation (2.1).

In the subsequent sections we will illustrate the propo-sed method in detail by applying it to a nonlinear evolution equation.

III. APPLICATION OF (G'/G)-EXPANSION METHOD FOR NONLINEAR HEAT CONDUCTION EQUATION

In this section, we will consider the following nonline-ar heat conduction equation [8]:

$$u_t - \alpha (u^n)_{xx} - u + u^n = 0 \tag{3.1}$$

In order to obtain the traveling wave solutions of Eq.(3.1), we suppose that

$$u(x,t) = u(\xi), \xi = kx + \omega t \tag{3.2}$$

k, ω are constants that to be determined later.

By using (3.2), (3.1) is converted into an ODE

$$\omega u' - ak^2(u^n)'' - u + u^n = 0 \tag{3.3}$$

Suppose that the solution of (3.3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i (\tfrac{G'}{G})^i \tag{3.4}$$

where a_i are constants.

Balancing the order of u' and u^n in Eq.(3.3), we have

$m+1 = mn+2 \Rightarrow m = -\frac{1}{n-1}$ .So we make a varia-ble $u = v^{-\frac{1}{n-1}}$, then (3.3) is converted into

$$\frac{\omega(n-1)}{3}(v^3)' + (n-1)^2 v^2 (v-1) + ak^2 n(2n-1)(v')^2 - ak^2 n(n-1)vv'' = 0 \tag{3.5}$$

Suppose that the solution of (3.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$v(\xi) = \sum_{i=0}^{l} b_i (\tfrac{G'}{G})^i \tag{3.6}$$

where b_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \tag{3.7}$$

where λ and μ are constants.

Balancing the order of $(v^3)'$ and vv'' in Eq.(3.5), we have $3l+1 = l+l+2 \Rightarrow l = 1$.So Eq.(3.6) can be rewritten as

$$v(\xi) = b_1(\tfrac{G'}{G}) + b_0, b_1 \neq 0 \tag{3.8}$$

b_1, b_0 are constants to be determined later.

Substituting (3.8) into (3.5) and collecting all the terms with the same power of $(\frac{G'}{G})$ together and equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$(\frac{G'}{G})^0 : -b_0^2 - n^2 b_0^2 - \omega n b_1 b_0^2 \mu + b_0^3 - 2nb_0^2 + \omega b_0^2 b_1 \mu$$
$$-ak^2 nb_1^2 \mu^2 - ak^2 n^2 b_1 b_0 \lambda\mu + ak^2 nb_1 b_0 \lambda\mu + 2ak^2 n^2 b_1^2 \mu^2 \quad + 2nb_0^3 + n^2 b_0^3 = 0$$

$$(\frac{G'}{G})^1 : 3ak^2 n^2 b_1 \lambda\mu - 2n^2 b_1 b_0 + 3b_0^2 b_1 + 6nb_0^2 b_1 - 4nb_1 b_0$$
$$+2ak^2 nb_1 b_0 \mu + 2\omega b_1 b_0^2 \mu + ak^2 nb_1 b_0 \lambda^2 - \omega nb_0^2 b_1 \lambda$$
$$+3n^2 b_0^2 b_1 - 2\omega nb_1 b_0^2 \mu - ak^2 n^2 b_1 b_0 \lambda^2 + \omega b_0^2 b_1 \lambda \quad -2ak^2 n^2 b_1 b_0 \mu - 2b_1 b_0 - ak^2 nb_1^2 \lambda\mu = 0$$

$$(\frac{G'}{G})^2 : -\omega b_1^3 n\mu + 3n^2 b_0 b_1^2 - 2nb_1^2 + 6nb_0^2 b_1 + 2ak^2 n^2 b_1^2 \mu$$

$$+3ak^2 nb_0 b_1 \lambda + \omega b_0^2 b_1 - b_1^2 - n^2 b_1^2 - 2\omega nb_0 b_1^2 \lambda + ak^2 n^2 b_1^2 \lambda^2$$

$$+3b_0 b_1^2 - \omega nb_0^2 b_1 - 3ak^2 n^2 b_0 b_1 \lambda + \omega b_1^3 \mu + 2\omega b_0 b_1^2 \lambda + 6nb_0 b_1^2 = 0$$

$$(\frac{G'}{G})^3 : a\omega b_1^3 \lambda - 2\omega nb_1^2 b_0 - \omega b_1^3 n\lambda + ak^2 nb_1^2 \lambda + b_1^3 + 2\omega b_0 b_1^2$$

$$2ak^2 nb_0 b_1 - 2ak^2 n^2 b_0 b_1 + ak^2 n^2 b_1^2 \lambda + 2nb_1^3 + n^2 b_1^3 = 0$$

$$(\frac{G'}{G})^4 : a\omega b_1^3 + ak^2 nb_1^2 - \omega b_1^3 n = 0$$

Solving the algebraic equations above, yields:

Case 1: when $\lambda^2 - 4\mu > 0$

$$b_1 = \pm\sqrt{\frac{1}{\lambda^2 - 4\mu}}, b_0 = \pm\frac{1}{2}\sqrt{\frac{1}{\lambda^2 - 4\mu}} + \frac{1}{2}$$

$$k = \pm\frac{n-1}{n}\sqrt{\frac{1}{a\lambda^2 - 4a\mu}}, \omega = \pm\frac{(n-1)\sqrt{\lambda^2 - 4\mu}}{n(\lambda^2 + 4\mu)} \tag{3.9}$$

Substituting (3.9) into (3.8), we have

$$v(\xi) = \pm\sqrt{\frac{1}{\lambda^2 - 4\mu}}(\frac{G'}{G}) \pm \frac{1}{2}\sqrt{\frac{1}{\lambda^2 - 4\mu}} + \frac{1}{2}$$

$$\xi = \pm\frac{n-1}{n}\sqrt{\frac{1}{a\lambda^2 - 4a\mu}}x \pm \frac{(n-1)\sqrt{\lambda^2 - 4\mu}}{n(\lambda^2 + 4\mu)}t \tag{3.10}$$

Substituting the general solutions of (3.7) into (3.10), we have:

$$v_1(\xi) = \mp\frac{\lambda}{2}\sqrt{\frac{1}{\lambda^2 - 4\mu}} \pm \frac{1}{2}(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}) \pm \frac{1}{2}\sqrt{\frac{1}{\lambda^2 - 4\mu}} + \frac{1}{2}$$

Then $u_1(\xi) = (v_1(\xi))^{-\frac{1}{n-1}}$

where $\xi = \pm\frac{n-1}{n}\sqrt{\frac{1}{a\lambda^2 - 4a\mu}}x \pm \frac{(n-1)\sqrt{\lambda^2 - 4\mu}}{n(\lambda^2 + 4\mu)}t$, C_1 and C_2 are two arbitrary constants.

Case 2: when $\lambda^2 - 4\mu < 0$

$$b_1 = \pm\sqrt{\frac{1}{4\mu - \lambda^2}}i, b_0 = \pm\frac{1}{2}\sqrt{\frac{1}{4\mu - \lambda^2}}i + \frac{1}{2}$$

$$k = \pm\frac{n-1}{n}\sqrt{\frac{1}{4a\mu - a\lambda^2}}i, \omega = \pm\frac{(n-1)\sqrt{4\mu - \lambda^2}i}{n(\lambda^2 + 4\mu)} \tag{3.11}$$

Substituting (3.9) into (3.8), we have

$$v(\xi) = \pm\sqrt{\frac{1}{4\mu - \lambda^2}}i(\frac{G'}{G}) \pm \frac{1}{2}\sqrt{\frac{1}{4\mu - \lambda^2}}i + \frac{1}{2}$$

$$\xi = \pm \frac{n-1}{n}\sqrt{\frac{1}{4a\mu - a\lambda^2}}ix \pm \frac{(n-1)\sqrt{4\mu-\lambda^2}\,i}{n(\lambda^2+4\mu)}t \qquad (3.12)$$

Substituting the general solutions of (3.7) into (3.12), we have:

$$v_2(\xi) = \mp\frac{\lambda}{2}\sqrt{\frac{1}{4\mu-\lambda^2}}i \pm \frac{1}{2}i\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right) \pm \frac{1}{2}\sqrt{\frac{1}{4\mu-\lambda^2}}i + \frac{1}{2}$$

Then $u_2(\xi) = (v_2(\xi))^{-\frac{1}{n-1}}$

where $\xi = \pm \frac{n-1}{n}\sqrt{\frac{1}{4a\mu - a\lambda^2}}ix \pm \frac{(n-1)\sqrt{4\mu-\lambda^2}\,i}{n(\lambda^2+4\mu)}t$, C_1 and C_2 are two arbitrary constants.

IV. CONCLUSION

The main points of the (G'/G)-expansion method are that assuming the solution of the ODE reduced by using the traveling wave variable as well as integrating can be expressed by an mth degree polynomial in (G'/G), where $G = G(\xi)$ is the general solutions of a second order LODE.

The positive integer m is determined by the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the reduced ODE, and the coefficients of the polynomial can be obtained by solving a set of simultaneous algebraic equations resulted from the process of using the method. Furthermore the method can also be used to many other nonlinear equations.

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